Objects & Proofs

Session 1. Neglect of Epistemic Considerations in the 20th century Logic (after Goran Sundholm)

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Plan of 3 Sessions :

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1. The Neglect of Epistemic Considerations in the 20-th century Logic ;

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3. Homotopy Type theory and Univalent Foundations of Mathematics.

Plan of Session 1

Pete Wolfendale

Reasoning is something that is *done*, and its something that can be done by processes other than us, processes that can and have been studied using reason, with the unforgiving precision of mathematical proof. Russells paradox and Gödels theorems lie at the beginning of an ongoing process through which we demonstrate reasons own limits, and then, following Turing, use these limits as purchase to pull it out of our hominid skulls and realise it in new and stranger forms.

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- Capacity to store and transmit knowledge through multiple generations of humans.
- Reliable and effective means of verification (proof) available to individual users (•).

To Sum Up :

There is good reason for philosopher to be an active player rather than only reactive observer (without disregarding the importance of the second thought !).

The role of philosophy does not reduce to fighting philosophical and other prejudices. New ideas are badly needed in new scientific technological developments. Ideas is a strong point of Philosophy (their implementation is not). Some old ideas in new forms and new contexts can do very good jobs.

JTB theory of knowledge : Plato

THEAETETUS : [Someone] said that true opinion, combined with reason, was knowledge, but that the opinion which had no reason was out of the sphere of knowledge ; and that things of which there is no rational account are not knowable— such was the singular expression which he used — and that things which have a reason or explanation are knowable.

SOCRATES : Which is probably correct — for how can there be knowledge apart from an account [logos = reason, definition] and true opinion ?

JTB theory of knowledge : Plato

SOCRATES : And yet there is one point in what has been said which does not quite satisfy me.

THEAETETUS : What was it ?

SOCRATES : What might seem to be the most ingenious notion of all : — That the elements or letters are unknown, but the combination or syllables known [Dream Theory].

JTB theory of knowledge : modern

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In short : Knowledge is **J**ustified **T**rue **B**elief.

Gettier Problem

Mary *believes* that a coin that she's about to toss will come up with *heads*. She *believes* it *because* her personal Daemon tells her so. Then Mary tosses the coin and it falls heads. This evidences that Mary's belief was *true*. It also has a justification of sort.

Does Mary's belief qualify as *knowledge*. JTB implies that the answer is positive. But arguably the correct answer is negative because the Daemon is Mary's phantasy or hallucination and she made the right guess by a mere chance/luck !

Knowing That, Knowing How and Knowing by Acquaintance

Beware that JTB is designed to cover ONLY propositional knowledge (aka knowledge-that) : to be truth-evaluated beliefs needs to be propositional !

Justification and Proof

The former concept may be more general than the latter. In what follows I talk about proofs and leave aside other forms of justification (if any).

JTB and KR

Even if JTB is not fully satisfactory as a general theory of propositional knowledge the distinction between true belief (true opinion) and knowledge, which dates back to Plato, is epistemically relevant. Surprisingly it is not used and even cannot be reconstructed in standard KR architectures. In CS/AI research/engineering "knowledge" stands for a sophisticated form of data representation and data proceeding that allows users to extract from the data a relevant information (according an individual demand) in a form that can be digested by humans, i.e., which is capable to produce true beliefs in human mind.

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A regular user of KR systems has no access to the raw data and the processing of those data from his perspective is epistemically opaque. If the reliance on the authority and good will (i) of institutions gathering and storing the raw data and (ii) of IT engineers building KR systems may qualify as a form of justification at all, such a justification is a very poor one ! (Compare the notion of "blind inference" below.)

Roots of the problem are philosophical as we shall now see. This is why Philosophy may contribute significantly to the future solution.

(A philosophical remedy that I rule out without further ado : trivializing knowledge at the extent it is equated to true belief or even to belief without qualification.)

Two Perspectives on Logic (Archbishop Whatley 1826)

LOGIC MAY BE CONSIDERED AS THE SCIENCE, AND ALSO AS THE ART, OF REASONING.

When reasoning we carry out acts of passage, “inferences”, from granted premises to novel conclusions. Logic is Science because it investigates the principles that govern reasoning and Logic is Art because it provides practical rules that may be obtained from those principles.

Sundholm on Reasoning

Reasoning is par excellence an epistemic matter, dependent on a judging agent. If the ultimate starting points for such a process of reasoning are items of knowledge, accordingly a chain of reasoning in the end brings us to novel knowledge.

Aristotle's Conception of Logic

Logic is grounded in Metaphysics aka the First Philosophy : Laws of Logic reflect Laws of Being.

The *Art* of Logic derives its rules from the *Science* of Logic, which in their turn reflect Principles of Being (= ontological principles)

Kant's Conception of Transcendental Logic

Science and Art of Logic need to be based on an a critical epistemological analysis of knowing (Erkenntnis) after examples of the best contemporary Science (Newton's Physics) — rather than on any sort of dogmatic speculative metaphysics.

Problem for Kant : his effort to follow his contemporary science was not sufficient as far as the pure mathematics was concerned (Lambert).

Hegel's Conception of Objective (Dialectical) Logic

Kant's and Aristotle's insights about the Science of Logic needs to be dialectically combined. Kant's subjective bias — including its transcendental version — needs to be rectified.

Problem for Hegel : apparently he simply abandoned a serious attempt to follow the contemporary developments in Science. As a matter of historical fact the Art of Logic based on Hegel's Science of Logic was more successfully used as a tool of political propaganda rather than a tool of reasoning in Science. Hence the general hostility toward Hegel on the part of science-oriented logicians and philosophers, which continues until today.

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- The authority of Kant and Kantianism as the mainstream Philosophy of Science has been challenged and rejected (Russell) by many on technical rather than conceptual grounds ; more traditional conceptions of logic re-established themselves (Bolzano) and were powered by new symbolic and mathematical means (Frege, Hilbert, Tarski).

What happened next ?

- As a result the epistemic dimension of logic has been largely neglected in the mainstream logic throughout the 20th century. Alternative conceptions of logic were largely marginalised because they were not supported by new technical (to wit mathematical) developments — with few notable exceptions (Kolmogorov, Martin-Löf, Girard).

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- The situation is getting better nowadays. More recent mathematical/logical developments are progressively digested by the philosophical community and a more profound philosophical discussion takes place in formal contexts. But a lot of outdated logical orthodoxy still remains operational under technical works. The philosophical openness and the technical excellence are EQUALLY essential for making a progress in this field !

Ontologies in KR

Consider in this context the fact that Formal Ontology today is a proper part of CS and IT (in KR) while methods of Formal Epistemology are generally underdeveloped and not applied in this field. This is a very strange situation in today's **Knowledge** Representation !

Symbolic calculus

comprises :

- an alphabet of symbols ;
- a set of words w_i built with the alphabet ;
- a set of rules r_i of form $w_1, \dots, w_k \vdash w$, which derive word w from given words $w_1, \dots, w_k$;
- set A (possibly empty) of axioms which are rules of special form $\vdash w$.

Propositional language : Syntax

Prop. language is a calculus with a distinguished finite set (alphabet) of symbols called *logical connectives*, which includes connective "$\rightarrow$" ; other symbols are called *propositional variables*.

Propositional theories : Syntax

is a set T of formulae closed under application of the standard *modus ponens* (MP) (other rules are allowed but not required).

Elements of T are called *theorems* of the given theory. The theory is called *axiomatic* when it comprises a distinguished subset $A \subset T$ of *axioms* such that all theorems of T are derivable from the axioms via applications of MP. The notion of derivation from a set Γ of hypotheses (denoted $\Gamma \vdash_T F$ or $\Gamma \vdash F$ when there is no risk of confusion) is standard.

Propositional theories : Semantics : Variables and Connectives

An *interpretation* of propositional theory is an assignment of to its propositional variables (in each formula) certain things called *semantic values*.

The standard semantics of Classical propositional logic involves two values called *truth-values*) : **Verum** (truth, $\top$) and **Falsum** (falsity, $\bot$).

Semantic (truth-) values of composed formulas built from variables with the available connectives is functionally ($=$ uniquely) determined by values of propositional variables (compositionality : truth-tables).

Propositional theories : Semantics : Variables and Connectives

The analogous truth-functional semantics for the Intuitionsitic prop. logic requires an infinite set of truth-values between the **Verum** and **Falsum**.

Propositional theories : Tautologies and Contradictions

Formulas that are true under all possible interpretations of variables are called (logical) *tautologies*. Example : LEM : $p \vee \neg p$ in Classical prop. logic. An axiomatic theory that generates all tautologies available in the given propositional language is called the (axiomatized) propositional *logic*. LEM is <u>not</u> universally valid in the Intuitionistic prop. logic.

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First-order language : Syntax

Idea : to unpack propositions.

Formally : an extension of a propositional language with :

- Predicate symbols and individual variables : $P(x_1, \ldots, x_n$;
- individual (non-logical) constants : $a, b, \ldots$;
- The existential quantifier $\exists$ and the universal quantifier $\forall$;
- additional deductive rules for quantified formulas : universal (existential) instantiation (generalisation)

First-order language : Semantics (very informally)

Multiple but there is the standard one called Set-theoretic aka Tarskian.

Idea : individual constants stand for certain fixed individuals, individual variables range over individuals (like variables range over numbers in algebra), predicate symbols stand for properties of and relations between the individuals ; quantifiers have their intuitive meaning. That makes makes first-order wwf's into propositions, which are truth-evaluated as above. The case when the resulting proposition is true is of special interest !

Example : formula $\forall x \exists y P(x, y)$ can be read as : for any given number n there exists a number m such that $m > n$.

First-order language : Semantics (slightly more formally)

Interpretation of a formula comprises

- set U called the *universe of discourse* ;
- assignment to individual constants certain fixed elements of U
- assignment to predicate symbol of arity k a subset of U^k, which intuitively consists of k-tuples of individuals that stand in the given relation or have a given property (for $k = 1$) ;
- treat bounded variables accordingly.

Providing a formula with some semantics along the above line is called its *interpretation*. An interpretation I that makes formula F into true proposition $[F]^I$ is called its *model*. It is also said that under the given interpretation the formula is *satisfied*. An interpretation that makes true all formulas of a given theory is called a model of this theory. Generally a formal theory admits multiple models.

Logical Inference and Formal Deduction

Let $A_1, \ldots, A_n, B$ are formulas and B is obtained from $A_1, \ldots, A_n$ via the available syntactic (deductive) rules. The fact that such a derivation exists is denoted as $A_1, \ldots, A_n \vdash B$. The idea is that conclusion $[B]$ logically follows from assumptions (aka premises) $[A_1], \ldots, [A_n]$. But how one can assure that the given syntactic rules of deduction indeed represent valid rules of logical inference ?

Logical Consequence

We want logical inferences to preserve the truthness of assumptions. Moreover, in order to distinguish logical inferences from other valid inferences we want logical inferences be purely *formal*, i.e., be valid independently of particular contents of the given assumption. In the given framework this translates into the following definition due to Tarski :

Conclusion $[B]$ follows *logically* from (is a semantic logical consequence of) assumptions $[A_1], \dots, [A_n]$ iff **all** interpretations that make true (satisfy) formulas $A_1, \dots, A_n$ also make true formula B ; in symbols $A_1, \dots, A_n \models B$.

Soundness : a MUST

if $A_1, \ldots, A_n \vdash B$ then $A_1, \ldots, A_n \models B$

Semantic Completeness : a desideratum

if $A_1, \ldots, A_n \models B$ then $A_1, \ldots, A_n \vdash B$

Facts (Gödel) :

- Classical Propositional and Predicate (First-Order) Logics are semantically complete.
- A first-order axiomatic theory that contains Arithmetic (i.e., a fragment of which can be interpreted as Arithmetic in the usual informal sense) is <u>not</u> semantically complete.

Problem :

Logical inference is not properly analysed in the above setting as an epistemic act. The key notion is that of (meta-theoretical) *relation* of semantic consequence ; syntactic derivations are designed to respect an ontological state of affairs assumed in advance ("all models").

Proof and Evidence :

A *proof* in this setting is understood as a purely syntactic object, viz. a chain of formulas built according to available rules. Such "proofs" are studied in a meta-theory (Hilbert's "meta-mathematics"), but their epistemic impact (if any) remains unclear.

Valid logical inferences construed as above do not constitute proofs. Example : the inference $ZFC+ \models FLT$, to the best of today's knowledge, is valid. However it is not a proof of Fermat's Last Theorem ! We *know* that the inference is valid because we have strong evidences supporting the claim, the crucial one being Andrew Wiles' informal proof !

To Sum Up

The above formal setting serves well as a tool for meta-mathematical study of certain aspects of logical reasoning and thus contributes to the Science of Logic. For example, the aforementioned Gödel's theorem are among indisputable achievements of this approach.

However it contributes only poorly to Logic as an Art. FOL and the associated deductive systems turn out to be rather useless as practical tools. Deductions in such systems, generally, are not feasible computationally.

Cassirer 1907 adversus Russell 1903

(a teaser for what follows in the remaining lectures)

"Here rises a problem that lies wholly outside the scope of "logistics" [= Formal Symbolic Logic]. All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions."

Cassirer 1907 adversus Russell 1903

"Thus the worry about laws governing the **world of objects** is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (E. Cassirer, Kant und die moderne Mathematik, 1907)

Thank you !