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Categories without Structures

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Plan

1. Renewing foundations (generalities)
2. Claim
3. Set theory, Category theory and Mathematical Structuralism
4. Categorification against Structuralism
5. Concluding Remarks
1. Renewing Foundations

Historical observation.

Foundations is the most dynamic part of mathematics. While the principle body of mathematical knowledge is the subject to continuing growth (progress), its foundations is a subject to continuing renewal. This is in adds with the architectural metaphor of science.

Example: a historical hermeneutics of Pythagorean theorem.

1. Euclid’s “Elements” 147

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle [equality in equicomposability]

2. Long & Maurrow 1997

Let XYZ be a right triangle with lengths of legs x and y, and hypotheuse of length z. Then

\[ x^2 + y^2 = z^2 \]

3. Dounced 1965

Two non-zero vectors \( x \) and \( y \) are orthogonal if

\[ (y - x) \cdot x = y \cdot x = 0 \]
While the theorem in (some) sense remains the same its foundations change dramatically. (In which sense? Do they share a common structure?)

**Big picture:**

The progress (= cumulative development) of science requires preservation of the earlier acquired knowledge. But it cannot be preserved in a frozen condition, it needs a permanent renewal. This is what foundations serve for. Renewal of foundations doesn't reduce to mechanical repetition: it is radical (pleonastically). There is no progress in foundations.

There is no eternal "absolute" foundations (albeit there are eternal mathematical truths)
2. Claim:

Structuralist foundations of mathematics supported the mainstream research throughout the 20th century.

Set-theoretic foundations of mathematics is a (controversial) version of Structuralist foundations. Cf. Bourbaki.

The idea of categorical (=category-theoretic) foundations of mathematics has emerged in the 1960-ies as an attempt to create a better vehicle for Mathematical Structuralism. However, categorical foundations have a potential to overcome Structuralism.

My present purpose is to outline this new development.

My ambition is to make philosophy of mathematics active rather than only reactive.

Slogan: The subject-matter of mathematics is covariant transformation, not invariant form!
3. Set theory, Category theory, and Mathematical Structureism

Maclane 1996 ("Structure in Mathematics")

"All infinite cyclic groups are isomorphic, but this infinite group [notice the switch to singular-R] appears over and over again—in number theory, in ornaments, in crystallography, in physics. Thus the "existence" of this group is really a many splendored matter. An ontological analysis of things simply called "mathematical objects" is likely to miss the real point of mathematical existence."

Cf. the case of natural numbers: they are even more promiscuous both internally and externally. What’s new?

V Proliferation of structures

V Free building of structures
**Official definition (Hellman)**

"Structuralism is a view about the subject-matter of mathematics according to which what matters are structural relationships in abstraction from the intrinsic nature of the related objects. [...] The items making up any particular system [sic!] exemplifying the structure in question are of no importance; all that matters is that they satisfy certain general conditions—typically spelled out in axioms defining the structure or structures of interest."

*G. Hilbert’s “Grundlagen” (1899) and his often-quoted letter to Frege. But... he is more explicit as to “exemplification” of structures:*

"One merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar..."


*There is no invariant form unless the transformation in question is reversible!*
(Infinite) sets are apparently ideal candidates for "items with no intrinsic nature" (extensionality). Hence "Cantor's Paradise". But...

Are axiomatic theories of sets structural in the same sense? Is the circularity involved vicious? \{\emptyset, \emptyset\} or \{\{\emptyset\}\}? Substance strikes back through a semi-naive conception of the cumulative hierarchy! (Skolem). Nedee Structuralist misgiving about sets:

N. Bourbaki (J. Dieudonné) 1950
("The Architecture of Mathematics, a footnote")

"We take here a naive point of view and do not deal with the thorny questions ..." Raised by the problem of the "nature" of the mathematical "beings" or "objects". Axiomatic studies of the 19th and 20th centuries have gradually replaced the initial pluralism of mental representations of these "beings" ... by an unitary concept,
Reducing all the mathematical notions (...) to the notion of set. This latter concept, considered for a long time as "primitive" or "undefinable" has been the object of an endless polemics (...). The difficulties did not disappear until the notion of set itself disappeared (...) in the light of the recent work on logical formalism. From this new point of view mathematical structures become, properly speaking, the only "objects" of mathematics.

A more precise point:
Lawvere 1965 ("Category of Categories as a Foundation for Mathematics")

"In the mathematical development of recent decades one sees clearly the rise of conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question then naturally arises whether one can give a foundation of mathematics which expresses wholeheartedly this conviction concerning what mathematics
is about, and in particular, in which classes and membership in classes do not play any role. Consider Zermelo’s choice of $\mathcal{E}$ as primitive. Under the semi-naive conception of sets it re-establishes their concreteness. Mind the fundamental role of $\emptyset$ ("nothing")?

Why categories may be helpful?

Barbola’s notion of mathematical object:
- a structured set. Metaphysics of Matter and Form. A structuralist intention is to get rid with the "matter" (sets).

Example: group $(G, \circledast)$
1. Underlying set $G$ (matter)
2. Group operation $\circledast$ (structure)

For all $a, b \in G$, $a \circledast b = c \in G$

$\circledast \subseteq G \times G \times G$

subject to axioms

Isomorphism of groups $(G, \circledast)$ and $(G', \circledast')$
1. $G \rightarrow G'$: set-theoretic bijection
2. if $a \rightarrow a'$, then $a \circledast b \rightarrow (a' \circledast' b')$
Not all groups are isomorphic.

Hence the talk of types of structures.

Homomorphism of groups:

1) A function (set-theoretically)
2) The same (really!)

Does \( h \) preserve structure?

\[
\begin{array}{c}
G_1 \\
(\, 1, 0 \,)
\end{array} \rightarrow \begin{array}{c}
G_2 \\
(1, 1)
\end{array}
\]

[\( 101 = 1 \)]

forgetful homomorphism

Elementary transformations are always "reversible" in this setting. For the notion of non-ordered pair is basic in set theory while the notion of ordered pair is not. Compare \( \{a, b\} \) and \( (a, b) = \{a, \{a, b\}\} \). The order is a structure. But think about this issue in Kratzen terms. Isn't temporal intuition equally basic? G. Russell in "Principles" (1903).

So there is a sense in which the notion of reversible transformation, i.e. isomorphism is basic in a set-theoretic setting while the notion of structure-preserving map is not.
**Categories**

**Examples**
- Sets and functions
- Groups and homomorphisms
- Topological spaces and continuous mappings
- Vector spaces and linear transformations

**Generally**
- **Objects and Morphism**
  
  **Axioms:**
  1) **composition**
  \[ f : A \rightarrow B, g : B \rightarrow C \quad \Rightarrow \quad \exists f \circ g : A \rightarrow C \]
  
  2) **identity:**
  \[ f \circ 1_A = f \]
  \[ 1_C \circ g = g \]

  3) **associativity**
  \[ f \circ (g \circ h) = (f \circ g) \circ h \]

**Remark:** Objects are redundant; they are identified with their identity morphisms. Only arrows!

**Def.**

A \( f : A \rightarrow B \) is isomorphism if

there exist \( \alpha : A \rightarrow B \) s.t.

1) \( f \circ \alpha = 1_A \)

2) \( \alpha \circ f = 1_B \)

In this setting the notion of general isomorphism is primitive while that of isomorphism is not!
Theorem (MacLane)
Any type of structure in Bourbaki’s sense forms a category.

Handling of mathematical (Bourbaki-style) structures with categories relies exclusively on their structural properties and ignores their “material” (set-theoretic) background. A systematic recovering of these structures by purely categorical means would apparently give a “purely structural” foundation of mathematics. This is done in 1963 by Lawvere: a categorical recovering of set theory through an axiomatization of the category of sets (up to the categorical equivalence) sets as “black boxes”. This is also Lawvere’s motivation in his work of 1965 on category of categories.

But...
4. Categorification against Structuralism

1) Large categories (of sets, groups, ...) are not structures in Brouwer's sense
   - size problem
   - no isomorphic copies

2) Taken as primitives morphisms lose their structural character (Hellman's critiques)
   Non-structure-preserving morphisms

a) Partial order: at most 1 morphism between objects
   isomorphic objects are identical

b) Group (categorically construed):
   one object, all morphisms
   cf. Erlangen

c) (non-trivial): Grothendieck topology
   for any small category
   covering families of incoming morphisms
   instead of covering families of open sets
   Conceptually: the distinction btw topological and mereological properties
3) Categorification (= taking morphisms and higher morphisms into account) and structural abstraction point to the opposite direction. The structural abstraction can be described as decategoriesification.

Baez & Dolan 1998 ("Categorification")

"The category FinSet, whose objects are finite sets and whose morphisms are functions is a categorification of the set N of natural numbers."

Long ago when shepherds wanted to see if the two herds of sheep were isomorphic they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in another. But one day along came a shepherd decategoriesification [= structural abstraction R²]. She realised one could take each herd and "count" it setting up an isomorphism between it and some set of numbers which were nonsense words like "one, two, three,..." specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism!"
In short, by decategorifying the category of finite sets, the set of natural numbers was invented.

According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habit, which we are now struggling to overcome by means of categorification.

4. \[ A \xrightarrow{f} B \]  In case \( f \) is a general (i.e. non-reversible) morphism there is no appropriate sense in which \( A \) and \( B \) can be called equivalent. So they cannot be abstracted away and replaced by their "invariant form."

The subject matter of mathematics is a covariant transformation, not an invariant form.

Morphism \( \mathcal{F} \xrightarrow{f} \mathcal{B} \) is covariant because it translates internal transformation of \( A \) into internal transformations of \( B \). So the covariance condition amounts to \( 1 \) connectedness, \( 2 \) non-well-founded. In other words, there is no absolutely constant element.
Remind that identities of \( A, B \) are contextually determined:
\[
\begin{array}{c}
\varepsilon \xrightarrow{1} f \\
\varepsilon \xrightarrow{1} g \\
\end{array}
\]
This makes the internal/external distinction equally "contextual." But what about the "God-given" logical identity? This is fixed with higher morphisms:
\[
\begin{array}{c}
f \xrightarrow{1} f \\
A \xrightarrow{1} A \\
\end{array}
\]
When \( A \) is iso this provides a structuralist framework: one "think up to an isomorphism" at a very basic level. But what if it is not?

**An absurd question:** \( 2 + 3 \rightarrow 5 \)

Is this operation reversible?

It depends:\n\[
\begin{align*}
5 \times 1 &= 4 \\
5 \times 3 &= 2
\end{align*}
\]
But 2 and 3 don't pass away forever after summing, do they?
The identity of \( f \rightarrow h \) can be always recovered in a larger category. But there is no category which recovers everything.

Scale relativity.
The evolution of Lawvere's views from the wholehearted structuralism in 1960-ies to...

1965 (Cat(Cat)) : a double foundation
1) First-order axiomatic treatment
2) "Internal definitions" of an (abstract) category as an object in CAT
   • 1 - terminal in CAT
   • 0 - initial in CAT
   • 0 \rightarrow 1
   • 1 \rightarrow A - object in A
   • 2 \rightarrow A - morphism in A

Is the first step redundant? G. "doing logic in topos". G. Skolodovskiy

1963

Functorial semantics. Models as functors
No categoricity in the old sense.

Commentary of 2003:
"The theory appears itself as a generic model"

Tarski (colloquial): "This man confuses mathematics and meta mathematics, syntax and semantics"
"A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and pursues a speculative "foundations" for its own sake is clearly a nonfoundation."

Notice the "historical "front"

"Structures are foundations" (a private talk)
5. Concluding remarks

1) Back to Pythagorean theorem:
   there are translation between its different
   versions, but these are, generally, not reversible.
   Older versions translate into later versions
   but, generally, not the other way round.
   Notice that (and existance of the backward
   translation doesn't imply the reversibility
   "unless the category in question is a pre-order").
   There is no "shared structure" for different
   versions of Pythagorean theorem. The history
   is not reversible!

2) Mathematical communities and individual
   mathematicians don't need to share a common
   structure of reasoning in order to communicate
   successfully. At least, not on a "micro-level".

3) Even if CT can provide a useful structuralist
   framework it also can (and in my view
   should) be used for a deeper revision of
   foundations of mathematics. The controversy
   between the Substance and the Form
   seems to be (for the moment) exhausted.