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Categories without Structures

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Plan

1. Renewing foundations (generalities)
2. Claim
3. Set theory, Category theory and Mathematical Structuralism
4. Categorification against Structuralism
5. Concluding Remarks

1. Renewing foundations

Historical observation:

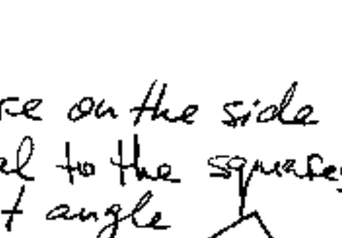
Foundations is the most dynamic part of mathematics. While the principle body of mathematical knowledge is the subject to continuing growth (progress), its foundations is a subject to continuing <u>renewal</u>. This is in odds with the architectural metaphor of science.

Example: a historical hermeneutics of Pythagorean theorem.

① <u>Euclid's "Elements" 1.47</u>

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle

[equality ~ equicomposability]



② *Long & Marrow 1997*

Let XYZ be a right triangle with lengths of legs x and y, and hypothemuse of length z. Then

$$x^2 + y^2 = z^2$$

③ Donnedu 1965

Two non-zero vectors x and y are orthogonal iff $(y-x)^2 = y^2 + x^2$

While the theorem in (some) sense remains the same its foundations change dramatically. (In which sense? Do 1-3 share a common structure?)

Big picture:

The progress (= cummulative development) of science requires preservation of the earlier acqured knowledge. But it cannot be preserved in a frozen condition, it needs a permanent renewal. This is what foundations serve for. Renewal of foundations doesn't reduce to mechanical repetition: it is radical (pleonastically). There is no progress in foundations.

There is no eternal „absolute" foundations (albeit there are eternal mathematical truths)

2. Claim:

Structuralist foundations of mathematics supported the mainstream research throughout 20th century.
Set-theoretic foundations of mathematics is a (controversial) version of Structuralist foundations. Cf. Bourbaki.
The idea of categorical (= category-theoretic) foundations of mathematics has emerged in 1960-ies as an attempt to create a better vehicle for Mathematical Structuralism
However categorical foundations have a potential to overcome Structuralism.

My present purpose is to outline this new development.
My ambition is to make philosophy of mathematics active rather than only reactive.

Slogan: The subject-matter of mathematics is covariant transformation, not invariant form!

3. Set theory, category theory, and Mathematical Structuralism

Maclane 1996 ("Structure in Mathematics")

"All infinite cyclic groups are isomorphic, but this infinite group [notice the switch to singular-AR] appears over and over again – in number theory, in ornaments, in crystallography, in physics. Thus the "existence" of this group is really a many splendored matter. An ontological analysis of things simply called "mathematical objects" is likely to miss the real point of mathematical existence."

Cf. the case of natural numbers: they are even more promiscuous both internally and externally. What's new?

- V Prolifiration of structures
- V Free building of structures

Platonic world

partaking
ideas
numbers
magnitudes
sensibilia

Plato contra Aristotle

there is no Substance only Form!

Official definition (Hellman)
"Structuralism is a view about the subject-matter of mathematics according to which what matters are structural relationships in abstraction from the intrinsic nature of the related objects. <...> The items making up any particular system [sic!] exemplifying the structure in question are of no importance; all that matters is that they satisfy certain general conditions - typically spelled out in axioms defining the structure or structures of interest"

Cf. Hilbert's "Grundlagen" (1899) and his often-quoted letter to Frege. But... he is more explicit as to „exemplification" of structures:
"One merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar..."

Cf. Awodey 1996: "The subject-matter of pure mathematics is invariant form". Cf. Erlangen
There is no invariant form unless the transformation in question is reversible!

(Infinite) sets are apparently ideal candidates for "items with no intrinsic nature" (extensionality). Hence "Cantor's Paradise". But...

Are axiomatic theories of sets structural in the same sense? Is the circularity involved vicious? $\{\emptyset, \{\emptyset\}\}$ or $\{\{\emptyset\}\}$? Substance strikes back through a semi-naive conception of the cummulative hierarchy! (Skolem). Hence Structuralist misgiving about sets:

N. Bourbaki (J. Dieudonné) 1950
("The Architecture of Mathematics, a footnote)
"We take here a naive point of view and do not deal with the thorny questions <...> raised by the problem of the "nature" of the mathematical "beings" or "objects". Axiomatic studies of the 19th and 20th centuries have gradually replaced the initial pluralism of mental representations of these "beings" <...> by an unitary concept,

reducing all the mathematical notions <...> to the notion of set. This latter concept, considered for a long time as „primitive" or „undefinable" has been the object of an endless polemics <...>. The difficulties did not disappear until the notion of set itself disappeared <...> in the light of the recent work on logical formalism. From this new point of view mathematical structures become, properly speaking, the only „objects" of mathematics."

A more precise point:

Lawvere 1965 ("Category of Categories as a Foundation for Mathematics")

"In the mathematical development of recent decades one sees clearly the rise of conviction that the relevant properties of mathematical objects are those which can be stated in terms of their <u>abstract structure</u> rather than in terms of the elements which the objects were thought to be made of. The question then naturally arises whether one can give a foundation of mathematics which expresses wholeheartedly this conviction concerning what mathematics

is about, and in particular, in which classes (9)
and <u>membership in classes</u> do not play any
role."

Consider Zermelo's choice of $\in$ as
primitive. Under the semi-naive conception
of sets it re-establishes their "concreteness".
Mind the fundamental role of $\emptyset$ ("nothing"?)

<u>Why categories may be helpful?</u>

Bourbaki's notion of mathematical object:
a structured set. Metaphysics of Matter
and Form. A Structuralist intention is to get
rid with the "matter" (sets).

Example: group $(G, \otimes)$

1. Underlying set G (matter)
2) Group operation $\otimes$ (structure)

For all $a, b \in G$, $a \otimes b = c \in G$

$\otimes \subset G \times G \times G$

subject to axioms

Isomorphism of groups $(G, \otimes)$ and $(G', \oplus)$

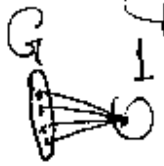
1) $G \xrightarrow{f} G'$: set-theoretic bijection
2) if $a \xrightarrow{f} a'$
$b \longrightarrow b'$ then $a \otimes b \xrightarrow{f} (a' \oplus b')$

But not <u>all</u> groups are isomorphic.
Hence the talk of <u>types</u> of structures.

Homomorphism of groups:

1) ~~G~~ h: function (set-theoretically)
2) the same

Does h (really!) „preserve structure"?



$(G,\otimes) \to (1,\circ)$ $\boxed{1\circ 1=1}$

forgetful homomorphism

Elementary transformations → are always "reversible" in this setting. For the notion of <u>non-ordered pair</u> is basic in Set theory while the notion of ordered pair is not. Compare $\{a,b\}$ and $\langle a,b\rangle=\{a,\{a,b\}\}$. The order is a structure. But think about this issue in Kantian terms. Isn't <u>temporal</u> intuition equally basic? Cf. Russell in „Principles" (1903)

So there is a sense in which the notion of reversible transformation, i.e. isomorphism is basic in a Set-theoretic setting while the notion of structure-preserving map is not.

Categories

Examples
- Sets and functions
- Groups and homomorphisms
- Topological spaces and continuous mappings
- Vector spaces and linear transformations

Generally: Objects and Morphism

Arrows:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h=fg"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

1) composition

$$\begin{tikzcd} {} \arrow[r, "f"] & A \arrow[loop above, "1_A"] \arrow[r, "g"] & {} \end{tikzcd}$$

2) identity: $f1_A = f$

$1_A g = g$

3) associativity

commutes $(fg)h = f(gh)$

Remark: Objects are redundant: they are identified with their identity morphisms. Only arrows!

Def $A \xrightarrow{f} B$ is isomorphism if

there exist $A \xleftarrow{g} B$ s.t. 1) $fg = 1_A$

2) $gf = 1_B$

In this setting the notion of general morphism is primitive while that of isomorphism is not!

Theorem (MacLane)
Any type of structure in Bourbaki's sense form a category.

Handling of mathematical (Bourbaki-style) structures with categories relies exclusively on their structural properties and ignores their "material" (set-theoretic) background. A systematic recovering of these structures by purely categorical means would apparently give a "purely structural" foundation of mathematics. This is done in 1963 by Lawvere: a categorical recovering of Set theory through an axiomatisation of "the" category of sets. (up to the categorical equivalence). Sets as "black boxes". This is also Lawvere's motivation in his work of 1965 on Category of categories.

But...

~~1 Large ca~~

4. Categorification against Structuralism

1) Large categories (of sets, groups, ...)
are not structures in Bourbaki's sense
- size problem
- no isomorphic copies

2) Taken as primitives morphisms lose (explicit) their structural character (Hellman's critiques)

Non-structure-preserving morphisms

a) Partial order: at most 1 morphism btw objects, isomorphic objects are identical

b) Group (categorically construed):
one object, all morphisms are isos

Cf. Erlangen

c) (non-trivial): Grothendieck topology for *any* small category
covering families of incoming morphisms instead of covering families of open sets.
Conceptually: the distinction btw topological and mereological properties.

3) Categorification (= taking morphisms – and higher morphisms – into account) and structural abstraction points to the opposite direction. The structural abstraction can be described as decategorification

Baez & Dolan 1998 („Categorification")

"The category FinSet, whose objects are finite sets and whose morphisms are functions is a categorification of the set N of natural numbers.
<...>
Long ago when shepherds wanted to see if the two herds of sheep were isomorphic they would look for an explicit isomorphism. In other words they would line up both herds and try to match each sheep in one herd with a sheep in another. But one day along came a shephard decategorification [= structural abstraction – AR]. She realised one could take each herd and "count" it setting up an isomorphism between it and some set of „numbers" which were nonsense words like "one, two, three, ..." „ specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitely establishing an isomorphism!

In short, by decategorifying the category
of finite sets, the set of natural numbers
was invented.

According to this parable, decategorification
started out as a stroke of mathematical genius
Only later did it become a matter of dumb
habit, which we are now struggling to
overcome by means of categorification."

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4. $A \xrightarrow{f} B$ In case f is a general (i.e non-reversible) morphism there is no appropriate sense in which A and B can be called equivalent. So they cannot be abstracted away and replaced by their "invariant form"

The subject-matter of mathematics is a <u>covariant</u> transformation, not an invariant form.

Morphism (functor) $A \xrightarrow{f} B$ is <u>covariant</u> because it translates internal transformation of A into internal transformations of B. So the covariance condition amounts to ① connectedness

Think of functions ② non-well-foundness.

pre-set-theoretically there is no absolutely "constant" elements

Remind that identities of A, B are contextually determined

$$\xrightarrow{f} \circlearrowright^{1_A} \xrightarrow[g]{} \qquad f1_A = f$$
$$1_A g = g$$

This makes the internal/external distinction equally "contextual"

But.. what about the "God-given" logical identity?

This is fixed with higher morphisms:

$$f1_A \xrightarrow{\alpha} f$$

$$\begin{tikzcd} \arrow[r, "f"] \arrow[rr, bend left] & A \arrow[r, "1_A"'] \arrow[u, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

When α is iso this provides a structuralist framework: one "think up to an isomorphism" at a very basic level.

But what if it is not?

An absurd question: $2+3 \to 5$

Is this operation reversible?

It depends.. $\begin{cases} 5 = 1+4 \\ 5-3 = 2 \end{cases}$

But 2 and 3 don't pass away forever after summing, do they?

The identity of $fg \to h$ can be always recovered in a <u>larger category</u>. But there is no category, which recovers everything

Scale relativity.

The evolution of Lawvere's views from the „wholehearted" structuralism in 1960-ies to...?

1965 (CatCat): a double foundation

1) First-order axiomatic treatment
2) „Internal definitions" of an (abstract) category as an object in CAT
   - $\mathbb{1}$ - terminal in CAT
   - 0 - initial in CAT
   - $0 \xrightarrow{2} \mathbb{1}$
   - $\mathbb{1} \xrightarrow{a} A$ - object in A
   - $2 \xrightarrow{f} A$ - morphism in A

   . . . . .

Is the first step redundant? Cf. „doing logic in topoi". Cf. Sketches.

1963

Functorial semantics. Models as functors
No categoricity in the old sense.
Commentary of 2003:
„The theory appears itself as a generic model"

Tarski (colloquial): "This man confuses mathematics and metamathematics, syntax and semantics"

Lawvere & Rosebrugh 2003

"A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A „pure" foundation that forgets this purpose and pursues a speculative „foundations" for its own sake is clearly a nonfoundation."

Notice the „historical" point.

"Structures are functors" (a private talk)

5. Concluding remarks

1) Back to Pythagorean theorem:
there are translation between its different versions but these are, generally, not reversible. Older versions translate into later versions but, generally, not the other way round. Notice that a mere existence of the backward translation doesn't imply the reversibility (unless the category in question is a pre-order). There is no "shared structure" for different versions of Phythagorean theorem. The History is not Reversible!

2) Mathematical communities and individual mathematicians don't need to share a common structure of reasoning in order to communicate successfully. At least, not on a "micro-level".

3) Even if CT can provide a usefull structuralist framework it also can (and in my view should) be used for a deeper revision of foundations of mathematics. The controversy between the Substance and the Form seems to be (for the moment) exhausted.