Constructive Axiomatic Method

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Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [..], an axiomatic theory [in the narrow sense of "axiomatic"] refers to a fixed system of things (or several such systems) [i.e. to one or several models].[..] This is an idealizing assumption that properly augments [?] the assumptions formulated in the axioms.

Euclid's Common Notions (Axioms) 1-3

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.

Euclid: Postulates 1-3

P1. Let it have been postulated to draw a straight-line from any point to any point.
P2. And to produce a finite straight-line continuously in a straight-line.
P3. And to draw a circle with any center and radius.

Remark

P1-3 are NOT propositions but (primitive) operations!

operation	input	output
P1	two points	segment
P2	segment	extended segment
P3	segment	circle

Shared Structure of Problems and Theorems: Proof by Construction

"Every Problem and every Theorem that is furnished with all its parts should contain the following elements:

- an enunciation
- an exposition
- a specification
- a construction [regulated by Postulates]
- a proof [based on Definitions, Hypotheses and Axioms]
- and a conclusion.

(Proclus, Commentary on Euclid, circa 450 A.D.)

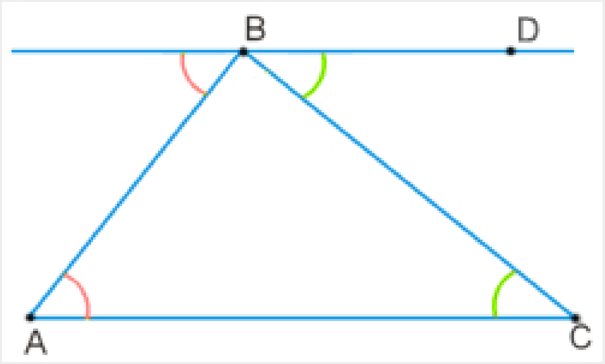
Proof by Construction

"Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Proof by Construction

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [..] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (Kant, Critique of Pure Reason, A 716 / B 744)

Interior Angles Sum Theorem



Hilbert 1899

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,..; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . [..]

Existential paraphrase of P1

Given two different point A, B there *exists* segment s having A, B as its endpoints.

Modal paraphrase of P2

Given two different point A, B it is always *possible* to produce segment s having A, B as its endpoints.

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- symbolic (syntactic) operations can be studied by the traditional genetic (constructive) mathematical method;
- thus the constructive method remains at work in formal *symbolic* axiomatic theories.

Hilbert on the role of symbolism

"No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction.

Hilbert on the role of symbolism

This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable." (Hilbert 1927)

Hilbert&Bernays 1934

"When we now approach the task of such an impossibility proof [= proof of consistency], we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.

Hilbert&Bernays 1934

[..] If we can conduct the impossibility proof without making any axiomatic-existential assumptions, should it then not be possible to provide a grounding for the whole of arithmetic directly in this way, whereby that impossibility proof would become entirely superfluous?"

Hilbert's answer is in negative because of his worries about infinities in Set theory and elsewhere in mathematics.

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- FAM does not suggest any systematic *method* of model-building;
- The last feature makes effective applications of FAM-based mathematics in natural sciences "unreasonable" (Wigner 1960)
- Up to the date the existential FAM has VERY limited application in the mainstream mathematical practice. It serves as a method of definition rather than method of proof (or presentation of proofs).

Cassirer contra Russell

"Thus the worry about laws governing the world of [empirical] objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (Cassirer 1907)

Thus there are strong reasons to develop genetic/constructive axiomatic methods in the modern context. How it may possibly work?

Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Natural deduction

Identity: $\overline{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}$(& - intro)

$\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Isomorphism

$$\& \equiv \times$$

$$\supset \equiv \longrightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A \& B$ is a pair consisting of a proof of A and a proof of B

Historical remark

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression "Curry-Howard isomorphism", which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.

Examples: Sets, Boolean algebras

Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- Objects: types / propositions
- Morphisms: terms / proofs

Lawvere’s philosophical motivation

- objective invariant structures vs. its subjective syntactical presentations
- objective logic vs. subjective logic (Hegel)

Internal Logic of Sets is Classical

The concept of CCC was discovered by Lawvere when he tried to axiomatize Set theory as a (first-order) theory of the category of sets (replacing $\in$ in its role of non-logical primitive by functions: ETCS.) This discovery marks Lawvere's shift from Hilbert to Euclid: instead of "using" the external (classical) FOL he now aims at building FOL internally as a part of his target axiomatic theory!

Higher-order generalization

- Quantifiers as adjoints to substitution; hyperdoctrines (Lawvere 1969)
- *Locally* Cartesian closed categories (LCCC) (Freyd 1972)

Topos theory

- Invention: Grothendieck circa 1960
- Axiomatization: Lawvere 1970

Toposes as generalized topological spaces

- Sheafs instead of opens
- Generalized coverings: Grothendieck topologies, sites

Internal Logic of Toposes

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos .

Lawvere on logic and geometry (continued)

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

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(Elementary) topos is a category which

- has finite limits
- is CCC
- has a subobject classifier

Simplification

Toposes are simpler than sets: one gets axioms for topos by relaxing ETCS axioms for sets!

Constructive Proof theory

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

Idea:

First- (and higher) order generalization of Curry-Howard

MLTT: key features

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- double interpretation of terms: elements of sets and proofs (witnesses) of propositions
- quantifiers: dependent types:
  disjoint sums and cartesian products of indexed families of sets

MLTT: a categorical model

MLTT is the internal language of LCCC (Seely 1983)

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: extensional versus intensional

- Extensionality: Propositional identity implies definitional identity (ex. LCCC)

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- Extensionality: Propositional identity implies definitional identity (ex. LCCC)
- First intensional (albeit 1-extensional) model: Hofmann & Streicher 1994:
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

Hofmann & Streicher groupoid model

judgement $\vdash A : type$ - groupoid A
judgement $\vdash x : A$) - object x of groupoid A type $Id_A(x, y)$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
(no reason to be empty unless x = y!)

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

HoTT: the idea

Types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory.

Fundamental group

Fundamental group G_T^0 of a topological space T:

- a base point P;
- loops through P (loops are circular paths $l : I \to T$);
- composition of the loops (up to homotopy only! - see below);
- identification of homotopic loops;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T:

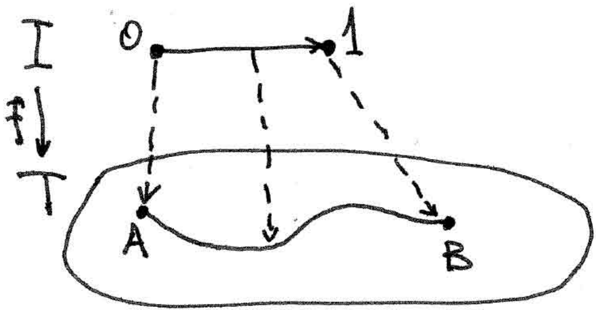
- all points of T (no arbitrary choice);
- paths between the points (embeddings $s : I \longrightarrow T$);
- composition of the *consecutive* paths (up to homotopy only! - see below);
- identification of homotopic paths;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T!

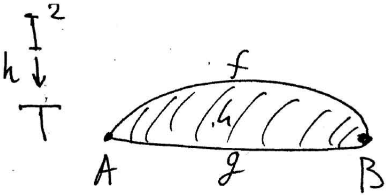
Path Homotopy and Higher Homotopies

$s : I \to T$, $p : I \to T$ where $I = [0, 1]$: paths in T
$h : I \times I \to T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \to T$: n-homotopy of $n-1$-homotopies h_0^{n-1}, h_1^{n-1} if $h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
Remark: Paths are zero-homotopies

Path Homotopy and Higher Homotopies



Homotopy categorically and Categories homotopically



$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- all points of T (no arbitrary choice);
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1}!

Composition of Paths

Concatenation of paths produces a map of the form $2I \to T$ but not of the form $I \to T$, i.e., not a path. We have the whole space of paths $I \to 2I$ to play with! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways!)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure!

Grothendieck Conjecture:

G^{ω}_{T} contains all relevant information about T; an omega-groupoid is a complete algebraic presentation of a topological space.

Homotopy model of MLTT

- Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). Extensionality one dimension up. (Streicher 1993).
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Axiom of Univalence

Homotopically equivalent types are (propositionally) identical. This means that the universe *TYPE* of homotopy types is construed like a homotopy type (and also modeled by ω-groupoid).
Axiom of Univalence is the only axiom of Univalent Foundations on the top of MLTT.

Voevodsky on Univalent Foundations

Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. [..]

Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose *syntax is tightly wedded to the intended semantical interpretation* in the world of everyday mathematics. In particular, it allows the <u>direct formalization</u> of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

h-levels

- (i) Given space is called A *contractible* (aka space of h-level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n + 1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-universe

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- Level 0: up to homotopy equivalence there is just one contractible space that we call "point" and denote pt;

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- Level 0: up to homotopy equivalence there is just one contractible space that we call "point" and denote pt;
- Level 1: up to homotopy equivalence there are two spaces here: the empty space $\emptyset$ and the point pt. (For $\emptyset$ condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call $\emptyset, pt$ *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that h-level n corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at h-level 1.

h-universe

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- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

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- Level 4: 2-groupoids

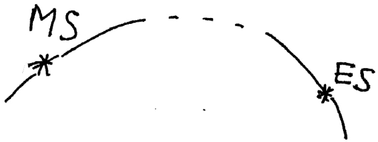
h-universe

- ..
- Level n+2: *n*-groupoids
- ..
- ω-groupoids
- ω-groupoids $(\omega + 1 = \omega)$

How it works

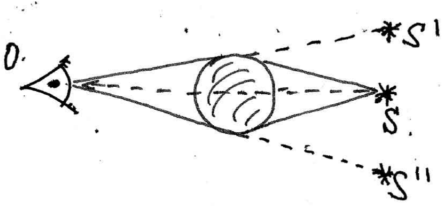
Let *iscontr*(A) and *isaprop*(A) be formally constructed types " A is contractible" and "A is a proposition" (for formal definitions see Voevodsky:2011, p. 8). Then one formally deduces (= further constructs according to the same general rules) types *isaprop*(*iscontr*(A)) and *isaprop*(*isaprop*(A)), which are non-empty and thus "hold true" for each type A; informally these latter types tell us that for all A "A is contractible" is a proposition and "A is a proposition" is again a proposition.

Naive stuff



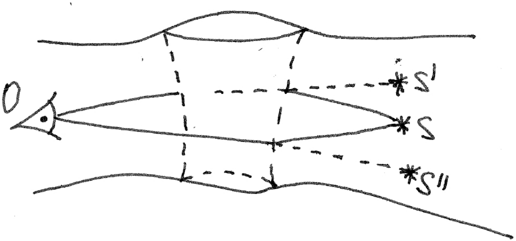
Identity through time

Naive stuff



Gravitational lensing

Naive stuff



Wormhole lensing

Serious stuff

Topos Physics:
A. Döring, Ch. Isham: 'What is a Thing?': Topos Theory in the Foundations of Physics (2008): http://arxiv.org/abs/0803.0417
Univalent Physics:
Urs Schreiber: Quantization via Linear homotopy types (Feb. 2014) http://arxiv.org/abs/1402.7041

Conclusion

The truly Modern Axiomatic Method is NOT Hilbert's Formal Axiomatic Method but
the Old Good Genetic Axiomatic Method of Euclid, Newton, Clausius, Lawvere and Voevodsky!

THANK YOU!