Renewing Foundations

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Abstract: A perpetual change of foundations observed in the real history of the discipline is not a historical accident but an essential feature of foundations. I distinguish between the progress of mathematics and renewal of its foundations and show how the latter contributes to the former with some historical examples. I also describe a mechanism of renewal of foundations, which has to do with needs of mathematics education, and provide an account of robustness of mathematical facts and arguments surviving through the change of their foundations. I outline my vision of today's situation and argue for the renewal of standard structuralist Bourbaki-style set-theoretic foundations in favor of new Category-theoretic foundations, which are linked to Structuralism historically and dialectically but imply a very different philosophical view on mathematics.

Key Words: foundation of mathematics, history of mathematics, category theory

1. Foundations and progress of mathematics

Foundations of mathematics, historically speaking, is the least stable part of this science. Pythagorean theorem, for example, stands firm since its early discovery while foundations of geometry (and in particular foundations of Pythagorean theorem) several times fall down and then have been totally renewed during the same historical period. Although these changes in foundations affected the theorem there is still a sense in which the Pythagorean theorem always remained the same. But foundations of mathematics underlying different versions of this theorem did not remain the same. The traditional view that mathematics is about number and magnitude and the modern view that mathematics is about sets and structures differ radically. Actually saying that different foundations differ radically is pleonastic since one describes a difference as "radical" exactly when it concerns foundations rather than anything else. The above observation shows that the popular architectural metaphor of science, which describes science as an edifice with a solid foundation, is completely misleading when one talks about science in a long-term historical perspective. The renewal of foundations is not only compatible with the progress of science but also helps to make this progress possible, as we shall now see.

The notion of progress in science assumes that once certain knowledge is acquired it later remains preserved and publicly available. It happens, of course, that certain beliefs, which at some point of history are generally seen as elements of current scientific knowledge, are later refuted and disqualified. However the notion of scientific progress concerns knowledge itself, not our current beliefs about what does and what does not qualify as knowledge. Blurring the difference between the true knowledge and the related beliefs would make the notion of progress incoherent. So let me now ignore the issue of belief revision and ask a different question: Where and how the ready-made knowledge is preserved and endures through human history?

According to Popper (1978) scientific knowledge and other products of human intellect live in a special metaphysical domain that he calls the Third World. The First world on Popper's account is that of physical processes and physical objects while the Second world is that of mental states. Popper's rationale behind his notion of Third Worlds is to avoid reducing knowledge to either mental states or physical processes:

Knowledge in the objective sense consists not of thought processes but of thought *contents*. It consists of the content of our linguistically formulated theories; of that content which can be, at least approximately, translated from one language into another. The objective thought content is that which remains invariant in a reasonably good translation. Or more realistically put: the objective thought content is what the translator tries to keep invariant, even though he may at times find this task impossibly difficult. (Italic is Popper's, underlining mine)

What is relevant to our present discussion here is not Popper's metaphysical argument but the way in which Popper thinks about thought contents in general and the content of scientific theories in particular. As a matter of course Popper doesn't identify the content with its linguistic expression. But he describes the content as an invariant of linguistic translations of a given expression from one language into another. Using today's popular mathematical jargon we can say that Popper thinks here of the thought content as a linguistic pattern taken "up to translation". In this sense the notion of linguistic expression still remains crucial in Popper's account.

I claim that Popper's notion of thought content fails to account for the long-term endurance of scientific, and in particular mathematical, knowledge. His theory better applies to the content of a religious doctrine rather than the scientific content. A teacher of religion may indeed translate a sacred text of his religion to his less educated pupils doing his best for keeping the original sense invariant. Even if the spirit of a religion generally doesn't reduce to its letter most developed religions use sacred texts as a means of preserving their identities through generations. But science proceeds very differently. A mathematical teacher - or at least a good mathematical teachers - doesn't try to transmit to her students the invariant content of some canonical text. Teaching the Pythagorean theorem today she doesn't "try to keep invariant" what Euclid has written about it some 2300 years ago but relies upon modern textbooks. If the notion of canonical text can make a sense in science at all it should be stressed that canonical scientific texts get quickly outdated, are revised, updated and periodically wholly rewritten. Euclid's *Elements* are often referred to as a typical example of canonical mathematical text. It is often said that until recently people used this book as a Bible of geometry. In fact this alleged dump habit never actually existed. To see this it is sufficient to look more precisely into book titled *Euclid's Elements*, which have been published before 19th century. One finds a surprisingly diverse literature under this title. Early publishers and translators of Euclid's *Elements* tried to produce a sound mathematical textbook rather than reproduce a canonical text. They didn't hesitate to improve on earlier editions of the *Elements* when they judged this appropriate. According to today's common standard the existing early editions of the *Elements* don't qualify as different versions of the same text. Any of these people could get today a copyright as the author of his *Euclid's Elements*. The notion of being an author is certainly changed since then.

Today's canonical Greek edition of Euclid's *Elements* has been produced by Heiberg and his assistant Menge only in the end of 19th century (see Euclides 1883-1886); noticeably these people were philologists, not mathematicians. So the idea to reproduce Euclid's text literally and translate it into modern languages "keeping its content invariant" is relatively recent; it is relevant to history of mathematics rather than to mathematics itself.

Euclid's book in its original form is no longer in use in schools not because some of Euclid's propositions have been judged false by education authorities. The fact that Euclid fails to meet today's standard of mathematical rigor is not the reason for it either because elementary textbooks anyway do not meet and are not supposed to meet such a standard. The principle reason why Euclid's *Elements* is no longer used in school is this: this book no longer provide a satisfactory basis for the further study of more advanced and more specific branches of mathematics. It perfectly did this job for quite a while but lost this capacity when mathematics essentially changed its shape. This dynamics is closely related to mathematical progress but it cannot be described itself as a progress. Kids learning mathematics are hardly cleverer today than they used to be a hundred years ago. Their learning capacities hardly essentially increased. But today's kids should be prepared to use and further develop mathematics that has been significantly progressed during the passed century. So they need a new curriculum.

This new curriculum cannot be just an extension of the older curriculum because this would require an increase of pupils' learning capacities. So they should study a different mathematics to begin with. This is why the change of the curricula does not qualify as progressive in the precise sense of the term. True, older and newer elementary mathematics textbooks typically share some of their contents. In particular older and newer textbooks equally include the Pythagorean theorem. Here is the Pythagorean theorem as it appears in Euclid's *Elements* (Proposition 1.47)

(1) In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

And here is how the same theorem appears in a modern textbook (Doneddu 1965, p.209)

(2) If two non-zero vectors x and y are orthogonal then $(y - x)^2 = y^2 + x^2$.

A care is needed in order to interpret the two propositions correctly. Euclid speaks here not about the areas of the squares but about the squares themselves: saying that the two smaller squares (taken together) are equal to the bigger square he means, roughly, that the bigger square can be composed out of pieces of the smaller squares. Minus on the left side of $(y - x)^2 = y^2 + x^2$ and plus on its right side don't stand for mutually inverse operations since the former operation applies to vectors while the latter applies to real numbers. Vectors, numbers and operations with these things are construed here as structured sets. In order to interpret not only the statement of the theorem in both cases but also its proof much more should be said about the corresponding theories. In particular, a lot is to be said about structuralist set-theoretic foundations of mathematics involved into Doneddu's book and foundations of Greek (and more specifically Euclid's) mathematics. The former issue is dealt with in what follows, the latter has been treated elsewhere (see my 2003 and further literature therein). But even without going into further details it is already clear that the difference between (1) and (2) is not that of linguistic surface. The two versions of Pythagorean theorem differ in their foundations, i.e. differ radically. And yet (1) and (2) express the same theorem.

What has been already said already suffices for showing that Popper's account of thought content doesn't apply to mathematics as far as this science is observed at large historical scales. Whatever the mathematical content might be it cannot be described as an invariant of linguistic translation; the notion of linguistic translation doesn't allow one to account for the long-term endurance of mathematical knowledge. The same is true for science in general. Unlike religious doctrines, poems, musical symphonies and some other inhabitants of Popper's Third World scientific knowledge endures in a long run through a permanent revision rather than a mere repetition of linguistic patterns or a mere retention of translational invariants as Popper suggests. Above I described above this revision as a pedagogical necessity. But it has also a philosophical aspect. I claim that the continuing questioning, revision and renewal of foundations is in fact a distinctive way in which science endures through time and performs a progress. This "unended quest" (to use Popper's word) concerns not only new yet unexplored domains of reality; it also concerns what is already known and well established. The renewal of foundations amounts to the dialectical refutation of older foundations and the dialectical positing of new foundations. This activity belongs to philosophy rather than to science itself. In this latter respect my view is traditional and qualifies as a form of foundationalism. But I also think that the notion of foundation does not make sense in abstraction from what it is (or supposed to be) foundation of. The historical performance of dialectically posited foundations crucially depends, in my view, on what scientists (including mathematicians) do with foundations. Thus my scientific foundationalism implies the need of a close cooperation between philosophy and science but definitely not the subordination of one of the two parties to the other. I subscribe to the following strong claim about the nature of scientific foundations:

> A foundation makes explicit the essential general features, ingredients, and operations of a science, as well as its origins and generals laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and pursues a speculative "foundations" for its own sake is clearly a nonfoundation. (Lavwere 2003)

2. What is next?

Today as ever foundations of mathematics need a radical renewal. The only serious candidate for future foundation of mathematics known to the date is a tentative category-theoretic foundation suggested by Lawvere (see his 1963) and other authors. I cannot present here my version of category-theoretic foundations systematically but I want to stress its feature, which seems me the most essential.

Set-theoretic foundations of mathematics suggest thinking of all mathematical objects as structured sets. A structured set has two basic components: a base set and a system of relations between elements of the base set determined by appropriate axioms. Think, for example, about an algebraic group construed as a set with a binary operation, which is subject to the well-known axioms. In this setting the notion of structure-preserving isomorphism (i.e. a reversible transformation) plays, roughly, the same role as the notion of equality plays in the more traditional mathematics: isomorphic structured sets are "equal" roughly in the same sense in which two copies of number 2 are equal. Sometimes this fact is expressed by saying that structured sets (for example, algebraic groups defined as above) are thought of up to isomorphism. A structured set thought of up to isomorphism is conventionally called a structure. Otherwise a structure can be described as an invariant of reversible transformations of structured sets. The philosophy of mathematics that describes mathematical objects as mathematical structures is known under the name of Mathematical Structuralism. We can see that Mathematical Structuralism perfectly accounts for set-based mathematics.

Category theory takes into account transformations (morphisms) of a more general type than isomorphism. Morphisms, generally, are non-reversible. Such a generalized notion of transformation naturally arises in the set-theoretic structuralist setting described above. Given the set-theoretic notion of group, for example, one may consider not only isomorphisms of groups but also group homomorphisms. The idea of category-theoretic foundations of mathematics amounts to making this generalized notion of transformation into a primitive and recovering the rest of mathematical universe on this basis. It has been shown, in particular, that Set theory can be reasonably recovered in these terms. In eyes of many people categorical mathematics is just a more refined version of structuralist mathematics. I disagree. One cannot think "up to homomorphism" in anything like the same way in which people think up to isomorphism doing structuralist mathematics. Non-reversible transformations unlike reversible ones, generally, have no invariants. So in the new setting one cannot, generally, qualify mathematical objects as structures. Instead of studying invariants of reversible transformations categorical mathematics studies transformations themselves. In order to accomplish the ongoing project of building practicable category-theoretic foundations of mathematics, further joint efforts of mathematicians and philosophers are needed.

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