From Sets to Topoi

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Categorical mathematics: Idea

Modern geometric philosophy holds firmly to the notion that the first thing one does after defining the objects of interest is to define the functions of interest. In our case the objects are Riemann surfaces, and we have already addressed complex-valued functions on Riemann surfaces. However functions are to be taken also in the sense of mappings between objects; once we de fine such mappings we will have a category of Riemann surfaces.
(Rick Miranda, *Algebraic Curves and Riemann Surfaces*, 1995)

Categorical Manifesto

[A]t the heart of 20th century mathematics lies one particular notion and that is the notion of a category.
Vladimir Voevodsky, 2002

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- composition of morphisms: if $Dom(g) = Cod(f)$ then there exists unique composite $h = f \circ g$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Mathematical Definition (continued)

For each object A *identity morphism* 1_A with the following properties:

(i) for each incoming morphism $\xrightarrow{f} A$ we have $f \circ 1_A = f$

(ii) for each outgoing morphism $A \xrightarrow{g}$ we have $1_A \circ g = g$

Mathematical Definition (continued)

Composition of morphism is associative: $(f \circ g) \circ h = f \circ (g \circ h)$

$$\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[drr] & C \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[urr] \arrow[rrr, "f \circ g \circ h"'] & & & D
\end{tikzcd}$$

the end of definition.

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- **Top**: topological spaces and continuous transformations (mind the definition!) ;
- **Grp**: groups and group homomorphisms;
- a group is a category with a single object and all morphisms isomorphisms;
- a poset is a category having at most one morphism btw two given objects.

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- TYPES: objects and morphisms (unnecessary)
- OPERATORS (takes morphisms to objects): *Dom* and *Cod*
- RELATION (on morphisms): $C(x, y; z)$ (composite)
- AXIOMS: (i) bookkeeping: if $C(x, y; z)$ then $Cod(x) = Dom(y)$ and $Dom(x) = Dom(z)$ and $Cod(y) = Cod(z)$; (ii) existence and uniqueness of composites, (iii) identity, (iv) associativity

Lawvere-style definition (1963)

The formalism of category theory is itself often presented in "geometric" terms. In fact, to give a category is to give a meaning to the word *morphism* and to the commutativity of diagrams like

$$A \xrightarrow{f} B$$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$

Lawvere-style (continued)

which involve morphisms, in such a way that the obvious associativity and identity conditions hold, as well as the condition that whenever

$$A \xrightarrow{f} B \;,\; B \xrightarrow{f} C$$

are commutative then there is just one h such that

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

is commutative. (*Adjoints in Foundations* Dialectica, 1969)

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- group homomorphisms (preserving);
- continuous transformations (reflecting);
- functors: morphisms of general categories, preserve *Dom*'s and *Cod*'s, identities and compositions;
- natural transformtions: morphisms of functors (in functor categories denoted (A, B) or B^A).

Adjunction

Adjoint situation:

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

for all objects X of A and objects Y of B

$$\begin{tikzcd}
Hom_A(FY, X) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "Hom_A(Fg,f)"'] & Hom_B(Y, GX) \arrow[d, "Hom_B(g,Gf)"] \\
Hom_A(FY', X') \arrow[r, leftrightarrow, "\sim"'] & Hom_B(Y', GX')
\end{tikzcd}$$

Adjunction (alternatively)

Adjoint situation:

$A \underset{g}{\overset{f}{\rightleftarrows}} B$ provided with natural transformations: $\alpha : A \longrightarrow fg$ and $\beta : gf \longrightarrow B$ such that $(g\alpha)(\beta f) = g$ and $(\alpha f)(g\beta) = f$:

$$\begin{tikzcd} g \arrow[r, "g\alpha"] \arrow[dr, "1_g"'] & gfg \arrow[d, "\beta g"] \\ & g \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f \arrow[r, "\alpha f"] \arrow[dr, "1_f"'] & fgf \arrow[d, "f\beta"] \\ & f \end{tikzcd}$$

Idea (Lawvere early 1960ies):

use functions instead of $\in$ (back to von Neumann 1930ies'); take an abstract category and specify it "into" the (?) category of sets

ETCS (1)

There is a singleton 1: $\forall S\ \exists!\ S \to 1$

Every pair of sets A, B has a product:

$$\forall T, f, g \text{ with } f : T \to A,\ g : T \to B,\ \exists!\ \langle f, g\rangle : T \to A \times B$$

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Every parallel pair of functions $f, g : A \to B$ has an equalizer:

$$\forall T, h \text{ with } fh = gh\ \exists!\ u : T \to E$$

$$\begin{tikzcd} T \arrow[d, "u"'] \arrow[dr, "h"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

ETCS (2)

There is a function set from each set A to each set B:

$$\forall\, C \text{ and } g: C\times A \to B,\ \exists!\ \widehat{g}: C \to B^A$$

$$\begin{tikzcd} C \arrow[d, "\widehat{g}"'] \\ B^A \end{tikzcd} \qquad \begin{tikzcd} C\times A \arrow[r, "g"] \arrow[d, "\widehat{g}\times 1_A"'] & B \\ B^A\times A \arrow[ur, "e"'] & \end{tikzcd}$$

There is a truth value $\mathit{true}: 1 \to 2$:

$$\forall\, A \text{ and monic } S \rightarrowtail A,\ \exists!\ \chi_i \text{ making } S \text{ an equalizer}$$

$$\begin{tikzcd} S \arrow[r, tail] & A \arrow[r, shift left, "\chi_i"] \arrow[r, shift right, "\mathit{true}_A"'] & 2 \end{tikzcd}$$

ETCS (3)

There is a natural number triple $\mathbb{N}, 0, s$:

$$\forall\ T \text{ and } x\colon 1 \to T \text{ and } f\colon T \to T,\ \exists!\ u\colon \mathbb{N} \to T$$

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\
& T \arrow[r, "f"'] & T
\end{tikzcd}$$

Extensionality: $\forall\ f \neq g\colon A \to B,\ \exists x\colon 1 \to A$ with $f(x) \neq g(x)$.

Non-triviality: $\exists\ \mathit{false}\colon 1 \to 2$ such that $\mathit{false} \neq \mathit{true}$.

Choice: $\forall$ onto function $f\colon A \to B,\ \exists\ h\colon B \to A$ such that $fh = 1_A$.

Classical case

Idea: **Set** is a logical framework (back to 19th c.: compare the Boolean algebra of classes/propositions)
From billing an axiomatic theory of sets within a fixed external framework to recovering the internal logic of **Set**.

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- the observation in 1934 by Curry that the types of the combinators could be seen as axiom-schemes for intuitionistic implicational logic.
- the observation in 1958 by Curry that Hilbert-style deduction systems, coincides on some fragment to the typed fragment of a standard model of computation known as combinatory logic,
- the observation in 1969 (published later in 1980) by Howard that the natural deduction can be directly interpreted in its intuitionistic version as typed lambda calculus.

Historical remark

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression "Curry-Howard isomorphism", which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Simply typed lambda calculus (type system for $\times$, $\longrightarrow$)

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \rightarrow T}$

$\dfrac{\Gamma \vdash t : U \rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Natural deduction (system for &, ⊃)

Identity: $\dfrac{}{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}$(& - intro)

$\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ (⊃-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ (⊃-elim aka *modus ponens*)

Curry-Howard Isomorphism

$$\& \equiv \times$$

$$\supset \equiv \longrightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A \& B$ is a pair consisting of a proof of A and a proof of B

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.

<u>Examples:</u> Sets, Boolean algebras

Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- Objects: types / propositions
- Morphisms: terms / proofs

Lawvere and Lambek 1969

A category is a deductive system in which the following equations hold between proofs:

- $f1_A = f, 1_B f = f, (hg)f = h(gf)$ for all $f : A \to B, g : B \to C, h : C \to D$
- $f = \bigcirc_A$ for all $f : A \to T$

Lawvere and Lambek 1969 (continued)

CCC:

- $\pi_{A,B} < f, g >= f$ and $\pi'_{A,B} < f, g >= g$
- $< \pi_{A,B}h, \pi'_{A,B}h >= h$ for all $f : C \rightarrow A, g : C \rightarrow B, h : C \rightarrow A \wedge B$
- $\varepsilon_{A,B} < h * \pi_{C,B}, h\pi'_{C,B} >= h$ and $\varepsilon_{A,B} < h * \pi_{C,B}, h\pi'_{C,B} >= h$ for all $h : C \wedge B \rightarrow A$ and $k : C \rightarrow B \Leftarrow A$ ($B \Leftarrow A$ aka B^A is implication)

Higher-order generalization: Hyperdoctrines (Lawvere)

- Quantifiers as adjoints to substitution; hyperdoctrines (1969)
- Toposes (1970)
- *Locally* Cartesian closed categories (LCCC) (1996)

Lawvere and Rosebrugh (2003) on the nature of Logic

The term "logic" has always had two meanings - a broader one and a narrower one:
(1) All the general laws about the movement of human thinking should ultimately be made explicit so that thinking can be a reliable instrument, but
(2) already Aristotle realized that one must start on that vast program with a more sharply defined subcase.

Lawvere and Rosebrugh (continued)

The achievements of this subprogram include the recognition of the necessity of making explicit
(a) a limited universe of discourse, as well as
(b) the correspondence assigning, to each adjective that is meaningful over a whole universe, the part of that universe where the adjective applies. This correspondence necessarily involves
(c) an attendant homomorphic relation between connectives (like and and or) that apply to the adjectives and corresponding operations (like intersection and union) that apply to the parts "named" by the adjectives.

Lawvere and Rosebrugh (continued)

When thinking is temporarily limited to only one universe, the universe as such need not be mentioned; however, thinking actually involves relationships between several universes. [..] Each suitable passage from one universe of discourse to another induces
(0) an operation of substitution in the inverse direction, applying to the adjectives meaningful over the second universe and yielding new adjectives meaningful over the first universe.

Lawvere and Rosebrugh (continued)

The same passage also induces two operations in the forward direction:
(1) one operation corresponds to the idea of the direct image of a part but is called “existential quantification” as it applies to the adjectives that name the parts;
(2) the other forward operation is called “universal quantification” on the adjectives and corresponds to a different geometrical operation on the parts of the first universe.

Lawvere and Rosebrugh (continued)

It is the study of the resulting algebra of parts of a universe of discourse and of these three transformations of parts between universes that we sometimes call "logic in the narrow sense". Presentations of algebraic structures for the purpose of calculation are always needed, but it is a serious mistake to confuse the arbitrary formulations of such presentations with the objective structure itself or to arbitrarily enshrine one choice of presentation as the notion of logical theory, thereby obscuring even the existence of the invariant mathematical content. In the long run it is best to try to bring the form of the subjective presentation paradigm as much as possible into harmony with the objective content of the objects to be presented; with the help of the categorical method we will be able to approach that goal.

Quantifiers as Adjoints

$X \xrightarrow{f} Y$ as "passage between universes" X, Y:
Suppose we have a one-place predicate (a property) P meaningful on set Y, so that there is a subset $P_Y \subseteq Y$ such that for all $y \in Y$ $P(y)$ is true just in case $y \in P_Y$. Using these data we can define a new predicate R on X as follows: we say that for all $x \in X$ $R(x)$ is true when $f(x) \in P_Y$ and false otherwise. So we get subset $R_X \subseteq X$ such that for all $x \in X$ $R(x)$ is true just in case $x \in R_X$. Thus we associate with every subset P_Y (every part of universe Y) a subset R_X and, correspondingly, a way to associate with every predicate P meaningful on Y a certain predicate R meaningful on X. Since subsets of given set Y form Boolean algebra $B(Y)$ we get a map between Boolean algebras (substitution map), notice the change of direction: $f^* : B(Y) \longrightarrow B(X)$

Quantifiers as Adjoints (continued)

The *left* adjoint to the substitution functor f^* is functor

$$\exists_f : B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of X) into $P \in B(Y)$ (subset of Y) consisting of elements $y \in Y$, such that *there exists* some $x \in R$ such that $y = f(x)$:

$$\exists_f(R) = \{y | \exists x (y = f(x) \wedge x \in R)\}$$

Quantifiers as Adjoints (continued)

The *right* adjoint to the substitution functor f^* is functor

$$\forall_f : \ B(X) \longrightarrow B(Y)$$

which sends every subset R of X into subset P of Y defined as follows:

$$\forall_f(R) = \{y | \forall x (y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$

Hyperdoctrines

A *hyperdoctrine* consists of a CCC T of "types" and functor h that associates (i) with every object A of T - a category $P(A)$ of "parts" of A, which in the given context are also thought of as "predicates" or "attributes"
Formally: $h : T^{op} \to C$ where C is 2-category
Canonical example: LCCC: C/A is CCC for all A of C

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[dl] \\ & A & \end{tikzcd}$$

Remark: connection to geometrical fibrations and fibered categories

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- Sites instead of (usual) part-based topologies; covering families of incoming morphisms (closed under composition and stable under pullbacks) instead of covering families of parts (open subsets).
- Grothendieck topos is the category of all sheaves on a site.

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- Observation: From the category-theoretic viewpoint toposes very much resemble sets! In order to capture Grothendieck toposes ETCS axioms need to be only slightly weakened.
- Intuition and Motivation: Thus we get categories of "continuously variable sets" instead of classical "static" sets (ETCS).
- Drawback: Grothendieck toposes are not caught in this way *precisely*: we also get (*elementary*) toposes, which are not Grothendieck (ex: finite sets)

Quantifiers and Sheaves (1970)

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos . We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry.

Elementary Topos:

- finite limits;
- CCC (terminal object, binary products, exponentials);
- subobject classifier

$$
\begin{tikzcd}
U \arrow[r, "!"] \arrow[d, hook, "p"'] & 1 \arrow[d, "true"] \\
X \arrow[r, "\chi U"'] & 2
\end{tikzcd}
$$

for all p there exists a unique χU that makes the square into a pullback

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- well-pointedness: for all $f, g : A \to B$, if for all $x : 1 \to A$ $xf = xg = y$ then $f = g$

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- NNO
- Choice: every epimorphism splits

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- Theorem: if T is topos and C is small then T^C is topos.
- Theorem (over-topos): if T is topos and $X \in T$ then T/X is topos.

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- For all terms s of type$B$ and all variables x of type A there exist term $(\lambda x)s$ of type B^A.

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Internal language of topos (aka internal semantics of the language)

- Each object A of topos T has a list of variables $x_1, x_2, ..$ over A; each variable over A is a term of type A;
- For any morphism $f : A \longrightarrow B$ and term s of type A sf is a term of type B;
- Morphism $c : 1 \longrightarrow A$ is constant term of type A;
- For all terms s_1 of type A and terms s_2 of type A there exist terms $< s_1, s_2 >$ of type $A \times B$;
- For all terms s of type$B$ and all variables x of type A there exist term $(\lambda x)s$ of type B^A.

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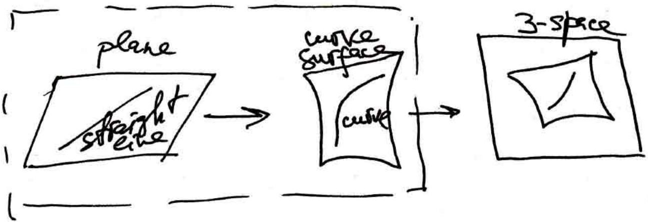
- External semantics (Kripke-Joyal) (satisfaction of formulas: truth)

The idea of intrinsic geometry (Gauss-Riemann-Klein)



Analogy with geometry

Compare the conceptual shift from Gauss' theory of curve surfaces to Riemann's general theory of (differentiable) manifolds: intrinsic construction of manifolds; no fixed ambient space is needed.

Analogy with geometry

Epistemically intrinsic and extrinsic properties of a given manifold are to be treated on equal footing. In the language of arrows the *intrinsic* properties are expressed by <u>incoming</u> morphisms while the extrinsic properties are expressed by <u>outgoing</u> morphisms (in particular, by embeddings into outer spaces). A given type/space is characterized by morphisms of both sorts.

Analogy with geometry

However there is a sense in which any given space can be fully characterized intrinsically! Is the outer logical framework always needed? Or we should rather think of categories of such frameworks? More technical work is needed to push the Riemanean point of view in Logic.

Claim:

Lawvere's axiomatization of topos theory is wholly intrinsic (even if its streamlining by McLarty et al. is not). We shall see that Voevodsky's axiomatization of Higher Homotopy theory (that is the basis of his proposed Univalent foundations of mathematics) shares the same feature.

THE END