All Objects are Arrows, All Arrows are Objects

Andrei Rodin

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Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method *introduces the objects* of a theory [..], an axiomatic theory [in the narrow sense of "axiomatic"] refers to a *fixed* system of things (or several such systems) [i.e. to one or several models].[..] This is an idealizing assumption that properly augments the assumptions formulated in the axioms.

Hilbert&Bernays 1934

When we now approach the task of such an impossibility proof [= proof of consistency], we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.

Hilbert&Bernays 1934

Yet, as a result of this deliberation, the following idea suggests itself right away: If we can conduct the impossibility proof without making any axiomatic-existential assumptions, should it then not be possible to provide a grounding for the whole of arithmetic directly in this way, whereby that impossibility proof would become entirely superfluous?

Hilbert's answer is in negative because of his worries about infinity.

Genetic aspects of FAM

Formulae-building, formal inferences. Genetic aspect is wholly syntactic. Formulae are objects of a meta-theory, not of the object-theory. A distinctive feature of FAM: limiting the constructive aspect of the method to syntax (in order to save infinities in semantics).

Some reasons to be dissatisfied with FAM

(1) FAM does not apply straightforwardly in the mainstream 20th c. maths. It serves for providing definitions rather than proofs. Example: Group theory is a model theory of the axiomatic group theory, i.e., the theory determined by the three group axioms. Where models come from?

Some reasons to be dissatisfied with FAM

(2) The impact of FAM on Set theory is unclear.
Example: The Independence of CH from ZF is well-established mathematical fact; the proof of this theorem (Gödel-Cohen) is <u>not</u> a formal axiomatic proof - notwithstanding the fact that this theorem treats a formal theory, namely ZF as its <u>object</u> (its subject-matter). This Independence result neither proves nor refutes CH. It does not allow to rule out CH as ill-posed either (after the example of Euclid's 5th Postulate). The full-scale relativism about mathematical statements is not consistent with the claim that the Independence of CH from ZF is well-established.

Some reasons to be dissatisfied with FAM

(3) The usual distinction between a theory and corresponding meta-theory doesn't make sense in the mathematical practice.

Some reasons to be dissatisfied with FAM

(4) The 20th c. showed no significant progress in the axiomatization of physics (Hilbert's 6th Problem). During this century FAM played no role at all in the mainstream research in physics and other natural sciences.
This one, in my view, is the strongest reason (however in my book I don't focus on it).

New-Old Genetic Axiomatic Method

Proof by (semantic) construction. Examples: (i) Euclid's geometrical proofs, (ii) Curry-Howard isomorphism and Categorical logic.
Categorical logic and its geometrical interpretations bring us back to Euclid, Newton and Clausius.

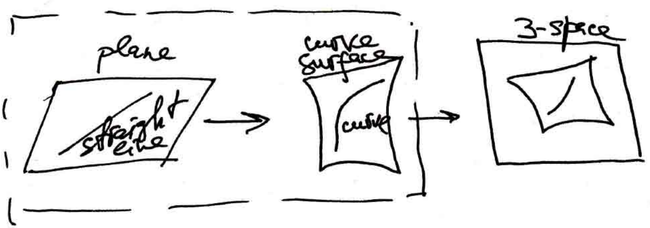
New-Old Genetic Axiomatic Method

Genetic Axiomatic Method is Object-Oriented Axiomatics Method. Compare the turn from Functional Programming to Object-Oriented Programming. The latter comprises the former (?) but is richer.

Claim

Renewal of Genetic Method is an essential part of Categorical logic in its historical development since late 1960ies. (See the case studies below.)

The idea of intrinsic geometry (Gauss-Riemann-Klein)



Objects are maps

General situation:

$$TYPE \xrightarrow{object} SPACE$$

Remarks:
Being a type and being a space are relational properties. Being an object is non-relational property.
Each object is of particular type and lives in a particular space.

Classical examples:

$$EPLANE \xrightarrow{eplane} ESPACE$$

$$\begin{tikzcd} CIRCLE \arrow[r, "circle_1"] \arrow[d, "circle_2"'] & EPLANE \arrow[dl, "eplane"] \\ ESPACE & \end{tikzcd}$$

Non-classical examples (19th century):

$$HPLANE \xrightarrow{pseudosphere} ESPACE$$

(Beltramy)

$$EPLANE \xrightarrow{horisphere} HSPACE$$

(Lobachevsky)
Remark: Pseudosphere and horisphere are not types/spaces but objects.

Objects are maps

Objects of the same type look differently in different spaces:

$$\begin{tikzcd}
 & \mathit{HSPACE} \\
\mathit{EPLANE} \arrow[ur, "\mathit{horisphere}"] \arrow[dr, "\mathit{eplane}"'] & \\
 & \mathit{ESPACE}
\end{tikzcd}$$

Objects of different types in the same space look always differently.

Remarks

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- (Historical) When it was understood that there is no unique representation space for all objects a popular reaction was to disregard the epistemic role of representation altogether and reduce objects to abstract individuals (possibly belonging to certain types). The true lesson of the 19th century geometry is the relativity of representation but not its epistemic insignificance.

Remarks

- (Historical) When it was understood that there is no unique representation space for all objects a popular reaction was to disregard the epistemic role of representation altogether and reduce objects to abstract individuals (possibly belonging to certain types). The true lesson of the 19th century geometry is the relativity of representation but not its epistemic insignificance.
- Different geometrical spaces are unified into a single whole through mutual mappings, i.e., through their shared objects. Objects link different spaces. This provides a geometrical unification of the 19th century geometry (as distinguished from its purely logical unification).

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- Objects (as maps) typically form categories;
- Sufficiently rich categories allow for internal logic (which reflects its object-building procedures);
- The internal logic of a given category is used for the axiomatic building of this very category;
- Thus Category theory (including Categorical logic) serves as a tool of renewed Genetic (object-oriented) Axiomatic Method.

Analogy with geometry

Compare the conceptual shift from Gauss' theory of curve surfaces to Riemann's general theory of (differentiable) manifolds: intrinsic construction of manifolds; no fixed ambient space is needed.

Analogy with geometry

Epistemically intrinsic and extrinsic properties of a given manifold are to be treated on equal footing. In the language of arrows the *intrinsic* properties are expressed by incoming morphisms while the extrinsic properties are expressed by outgoing morphisms (in particular, by embeddings into outer spaces). A given type/space is characterized by morphisms of both sorts.

Analogy with geometry

However there is a sense in which any given space can be fully characterized intrinsically. In that sense the Euclidean Planimetry fully describes EPLANE as a space. Extrinsic properties of EPLANE reveal themselves when the EPLANE embeds into ESPACE, HSPACE, etc.

Traditional essentialism requires to fix intrinsic properties first and study extrinsic (relational) properties afterwards. I do *not* endorse this view.

Claim

Lawere's axiomatization of Topos theory and Voevodsky's axiomatization of Higher Homotopy apply NAM rather than FAM.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.

Historical remark

Foundational consideration played a crucial role throughout the history of the subject (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression "Curry-Howard isomorphism", which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere's philosophical motivation

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- objective invariant structures vs. its subjective syntactical presentations
- objective logic vs. subjective logic (Hegel)

Internalization of Logic via Axiomatization of Set theory

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- 1969: Cartesian Closed Categories (CCC)
- 1969-70: Quantifiers as adjoints to substitution; hyperdoctrines
- 1970: Toposes

Lawvere on logic and geometry

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos .

Lawvere on logic and geometry (continued)

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

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(Elementary) topos is a category which

- has finite limits
- is CCC
- has a subobject classifier

Claim:

Lawvere's axiomatic theory of (elementary toposes) is not built by FAM. Instead of rebuilding topos theory with a pre-established logical framework Lawvere reveals the logical aspect of toposes. This move brings about a (genetic) axiomatic theory!

Pre-history:

For deductions over X, one may take provable entailments [..] or one may take suitable "homotopy classes" of deductions in the usual sense. One can write down an inductive definition of the "homotopy" relation, but the author does not understand well what results. (Lawvere: *Equality in hyperdoctrines* 1970)

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- double interpretation of types: "sets" and propositions
- double interpretation of terms: elements of sets and proofs of propositions
- higher orders: dependent types (sums and products of families of sets)
- higher identity types (in non-extensional versions)
- MLTT is the internal language of LCCC (Seely 1983)

Homotopy Type theory and Univalent Foundations

- Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). Extensionality one dimension up. (Streicher 1993).
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Voevodsky on Univalent Foundations

The broad motivation behind univalent foundations is a desire to have a system in which mathematics can be formalized in a manner which is as natural as possible. Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless.

Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose syntax is tightly wedded to the intended semantical interpretation in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

h-levels

- (i) Given space is called *A contractible* (aka space of h-level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

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- Level 1: up to homotopy equivalence there are two spaces here: the empty space $\emptyset$ and the point pt. (For $\emptyset$ condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call $\emptyset, pt$ *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that *h*-level n corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at *h*-level 1.

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- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

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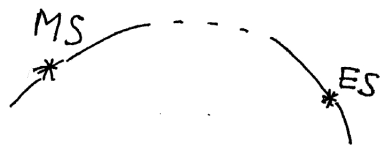
- Level 4: 2-groupoids

h-universe

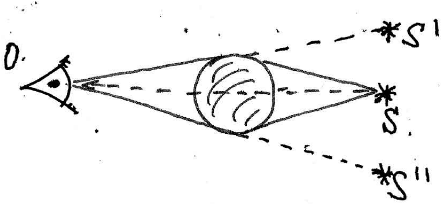
- ..
- Level n+2: *n*-groupoids
- ..
- ω-groupoids
- ω-groupoids $(\omega + 1 = \omega)$

Possible application: physical object-building

Identity through time

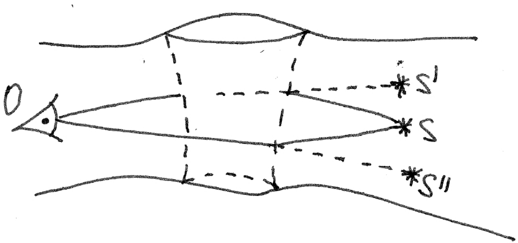


Possible application: physical object-building



Gravitational lensing

Possible application: physical object-building



Wormhole lensing

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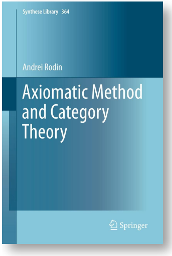

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A. Rodin, State University of Saint-Petersburg, Saint-Petersburg, Russian Federation

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http://arxiv.org/abs/1210.1478

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