Categorical Model Theory and the Semantic View of Theories

Andrei Rodin

Smirnov Readings 2017

Moscow, 16 June 2017

(Once) Received view

A theory is a set of propositions (expressed in a formal language L). It can be possibly (but not generally) represented through a list of *axioms* and described as the deductive closure of these axioms (Carnap).

Arguably (P. Suppes, B. van Fraassen et al.) a typical scientific theory is not a system of proposition : the "Non-Statement View of Theories"

Semanitc view

A theory is a class of models but not an axiom system, nor its deductive closure. (Suppes, Sneed, Stegmüller, Balzer, Moulines, van Fraassen)

Suppes 2002: Term "model" is used in logic and science similarly. One and the same theory may allow for many different axiomatizations.

Implementation

A Bourbaki-style set-based formalism:

Signature: the list of all non-logical terms of the given theory; abstract set-theoretic models of the given signature.

Example: Group theory

$< G, \otimes >$ where G is a set and $\otimes$ a binary operation (=ternary relation on G^3)

Axioms (expressible in CFOL): (i) $\otimes$ is associative; (ii) existence of unit and (iii) inverse elements

Practical outcome (in formalizing science for scientific needs): poor

Hans Halvorson

"Scientific Theories" (forthcoming). The controversy between syntactic and semantics approaches is artificial. ... Suppes' Bourbaki-style semantic presentation of theories is language-dependent and gives little or no advantage. The "semantic view" arguments is are no longer relevant in the presence of formal theories of semantics.

Hilbert-style and Gentzen-style axiomatic presentation

- Hilbert: many axioms and a single rule (MP). Ex.: ZF, Tarski's Elementary Geometry and similar
- Gentzen: many rules and few (possibly no) axioms (= rules with the empty assumption). Ex. Natural Deduction, Euclid's geometry, HoTT

Idea:

To use the G-style systems for non-logical theories (by allowing for non-logical rules).

Desiderata for a formal framework for Physics

- support deduction from first principles (first elements), including non-propositional ones (primitive objects, types, etc.);
- combine logical rules with constructive rules for non-propositional objects;
- support thought-experimentation and experimental design.

Constructive axiomatic method (arXiv:1408.3591 forthcoming in Logique et Analyse)

Theories satisfying the above desiderata I shall call *constructive* axiomatic theories.

An additional reason to prefer the Gentzen style: computability. Ex.: Cubical Type theory (Univalence Axiom is replaced by new rules).

Constructive Model theory informally

"Low-level" physical models of constructive theories are concrete *methods* for conducting particular experiments (Fraassen's "experimental design") and making observations.

Even if the sole purpose of experiments and observation is to give yes-no answers to certain questions an experiment and an observation need to be *designed*. Experimental design cannot be effective if it is done only by trials and errors.

Hintikka 2011 on the (standard) Axiomatic Method

"The class of structures that the axioms are calculated to capture can be either given by intuition, freely chosen or else introduced by experience."

Thought-experimentation without following certain adopted rules (which are <u>not</u> fixed once and for all) is trivial and does not meet scientific needs. Material experimentation is always rule-based, which allows experiments to be reproducible.

Tarski-style Model theory

Model $\models$ *Theory*
in words: The Model *satisfies* (= makes true) the corresponding Theory.

Interpretation : *Signature* $\rightarrow$ *Structure*
where a *signature* is a list of non-logical constants (provided with an additional information identifying each symbol as either an individual constant, or a function constant or a relation constant).

Assumptions 1-2

- A theory is a system of formal sentences (= sentential forms), which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms.

Two distinct points of a straight line completely determine that line

If different points A,B belong to straight line a and to straight line b then a is identical to b.

Assumption 3

Structures are *set-theoretic* structures.

Lawvere 1963: Functorial Semantics of Algebraic Theories

Idea: use categories instead of signatures (thus blurring the distinction btw. logical and non-logical terms)

Algebraic (Lawvere) theory: category LT with finite products and distinguished object X s.th. every object A in C is isomorphic to X^n for some finite number n.

Model: $LT \rightarrow SET$ that preserves finite limits.
Generalized Models: $LT \rightarrow C$ where C has finite limits.

Sketches (Ehresmann, Bar et al.

Observation: Even if (small) category C does not have (co)limits the presheaf category $\hat{C} = [C, SET]$ does. This allows for using *sketches* "instead of" theories.

Syntactic aka Classifying aka "Walking" Categories

Idea: a category "freely generated from the given syntax"

contexts are objects, substitutions of variables are morphisms

Generic Models

Universal property: *Synt*(T) is *initial* in *Mod*(T)

(Elephant D1.4, Th. 1.4.6)

Internal Language

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

$$Model : T \to Lang(C)$$

(in Theories)

Problem:

It is not clear whether Tarski's notion of model based on the satisfaction relation and his T-schema covers the functorial notion(s) of model in all cases. Categorical model theory may need an independent philosophical underpinning.

Claim:

Existing models of Homotopy Type theory are not Tarskian models and cannot be described in terms of the satisfaction relation and the T-schema.

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- ...

HoTT

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of interpretations? NOT just that.

The homotopical semantics of MLTT, which is used in HoTT, is not compatible with the informal semantics of MLTT proposed by Martin-Löf in 1983!

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of *h*-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of *h*-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of *h*-level n.

h-hierarchy

(-2) single point *pt*;

(-1) the empty space $\emptyset$ and the point *pt* : truth values aka *classical* or "mere" propositions

(0) sets (discrete point spaces)

(1) (flat) path groupoids but no non-contractibe surfaces

(2) 2-groupoids (paths and surfaces but no non-contractible volumes)

-
-

(n) n-groupoids

- ...

(ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does not take this structure into account. HoTT semantics does.

Competing approaches to modeling HoTT (an open controversy

- Awodey: classifying categories, natural models
- Voevodsky: (1) an interpretation of rules MLTT in the category of simplicial sets (2009);
  (2) contextual categories (Cartmell), C - systems, the Initiality Conjecture stands open.

Conclusions:

- A theory is, generally, not a system of propositions; it is rather a system of rules applied to propositions along with some non-propositional objects. Examples : Euclid, HoTT , Newton
- Functorial models are not, generally, Tarskian models - as evidenced by HoTT and its known models. In functorial models the distinction between logical and non-logical terms does not apply in its usual form. The generalized satisfaction relation between a constructive theory and its models is expressed in terms of *rule-following* (technically expressed as functoriality) rather than truth-evaluation. It does not reduce to the standard Tarskian notion of satisfaction for sentences at the pain of Carrol paradox.

Conclusions:

- The Non-Statement aka Semantic view of theories is not appropriately captured by the standard methods of formal semantics. The proposed concept of Constructive Axiomatic theory and its model (motivated by HoTT and its model theory) is a viable alternative.

Further Research

- A further study of (limits of) translations between Hilbert-style and Gentzen-style axiomatic presentations;
- Knowledge-that and knowledge-how;
- Formal reconstruction of physical and other scientific theories in a constructive axiomatic setting.

THANK YOU!