Identity, Abstraction, and (De)Categorification

1. Identity, Equality, and Equivalence

Symbol "=" in expressions like 3=3 prima facie allows for two different interpretations: it may be read (i) as identity or (ii) as relation of equality between different *copies* (tokens, "doubles") of 3. Considering this ambiguity in his *Grundlagen* Frege opts for (i) and disproves usual way of thinking about numbers as existing in multiple copies. Geometrical examples show that such straightforward identification of mathematical equality with logical identity may not work in different contexts. For in geometry unlike arithmetic term *equality* may mean - and did mean in the history - different things. Euclid uses the term "equality" (το ισον) of plane geometrical figures in the sense of equicompositionality but there are obviously other equivalence relations in geometry which may be equally considered as "substitutes of identity" for example congruence, (geometric) similarity, and affinity. There is a sense in which the "same figure" means the same shape and the same size, and there is another sense in which the "same figure" means only the same shape, and the "same shape" can be also specified differently. In addition geometry unlike arithmetic allows for identification of its objects by direct naming, usually through naming of its most important points. This allows us to distinguish two different triangles ABC and A'B'C' which are "same" in any of above senses. There is apparently no clear argument which would allow us to chose one of these senses of "same" as basic and eliminate others as the abuse of the language. So the situation in geometry (even classical!) is exactly as Manin (2002) describes it for a different purpose:

[T]here is no equality in mathematics objects, only equivalences.

2. Definitions by Abstraction

To push forward his project of reducing plural colloquial meanings of "same" in mathematics to a standard notion of identity provided by logic Frege proposed the method of "definition by abstraction". Frege's principle example of such definition is geometrical:

The judgment "line *a* is parallel to line *b*", or, using symbols *a//b* , *can be taken as identity* [italic mine - AR]. If we do this, we obtain the concept of direction, and say: "the direction of line *a* is identical with the direction of line *b*". Thus we replace the symbol // by the more generic symbol =

[denoting identity - AR], through removing what is specific in the content of the former and dividing it between *a* and *b*. [1]

The definition by abstraction allows for reduction of multiple colloquial meanings of "same" in mathematics to universal logical identity concept through introduction of new abstract objects. This method is problematic from logical point of view (Frege himself finally rejects it) but I want to stress a different point : it does not provide what a mathematician is normally looking for. Frege's "direction" (not to be confused with orientation!) is hardly an interesting mathematical object; this notion might play at most an auxiliary role in geometry and can be easily dispensed with. A *family* of parallel lines (in other words - a line defined *up to* parallel translation) apparently can make the same job as Frege's abstract direction but the former is more convenient for a mathematician. Similarly it is more convenient to think of natural number as a family of equal doubles rather than as an unique abstract object. Frege's definition by abstraction does not provide abstraction in a mathematically interesting sense. A mathematician who is not interested in logical regimentation of his discipline in Frege's vein does not need definitions by abstraction either. But it remains up to such mathematician to explain from logical point of view words "family" and "up to" used in mathematical contexts.

3. Relations versus Transformations

Replacement of given equivalence xEy by identity x=y proposed by Frege allows for a stronger interpretation than "abstraction". Namely, E can be interpreted as *reversible transformation,* which turns x to y and the other way round, and the identity = - as identity *through* this transformation. In the case of congruence the corresponding transformation is (Euclidean) *motion*: y is the *same* object x but moved up along Euclidean plane in a certain way. x and y are said here to be *same* in the same sense in which, for example, me yesterday and me today is the same person. So we think here about given mathematical object as a *substance* capable for changing its states or positions.

This substantialist reinterpretation of mathematical relations may look like an exercise in old-fashioned metaphysics but surprisingly it appears to be very fruitful from the mathematical point of view. Indeed the language of transformations is not formally equivalent to that of relations as one might expect but is richer. Given relation xRy there are, generally speaking many well-distinguishable transformations turning x into y while xRy only says that there

[1] Frege (1964). p. 74e.

exists one. Moreover reversible transformations (of same object) form a certain structure called *group*. This fact remains completely hidden when one reduces transformations to relations. To see the difference consider the following table:

language of relations	language of transformations
we write x≈y for "set x is equivalent (isomorphic) to set y"	we write f: X→ X or simply f for an isomorphism from set X to itself (automorphism)
≈ is equivalency relation; this means that:	automorphisms of X form a group; this means that:
≈ is transitive: x≈y and y≈z implies x≈z.	given automorphisms f, g there exists unique automorphism fg resulting from application of g after f.
≈ is reflexive: every set x is isomorphic to itself: x≈x	there exist identity automorphism 1 such that 1f=f1=f for any f
≈ is symmetric: if x≈y then y≈x	for every atomorphism f there exists its invers f^{-1} such that $f f^{-1} = f^{-1}f = 1$

Here basic facts about the (standard) equivalence relation between sets listed in the left column are translated into the language of transformations in the right column. The following notes explain how this translation works:

<u>line 1</u>: Sets x, y from the left column are identified in the right column as explained above. Notice that x≈y is a *proposition* while f: X→X is a thing, namely a map (function, isomorphism) from X to itself. Proposition x≈y *says that* there exists an isomorphism from x to y, while f *is* such an isomorphism. Things contain more information about themselves than mere evidences of their existence but they often cannot speak themselves. So translation in this line is not wholly transparent in either direction. Morphisms contain more information than the corresponding relation but this information lacks a well-defined logical form.
<u>line 3</u>: Transitivity of ≈ does not wholly grasp the concept of composition of isomorphisms, in particular it doesn't grasp the fact that the composition is unique.

<u>line 4</u>: The reflexivity of ≈ does not grasp at all the concept of identity isomorphism, so the "translation" in this line is superficial and even misleading. For the reflexivity doesn't imply the distinction between identity isomorphisms and other isomorphisms.

<u>line 5:</u> The reversibility of isomorphisms is not wholly grasped by the fact that ≈ is symmetric (notice the reference to the identity in the right column and see the comment to the previous line).

Does the approach outlined above provide any viable alternative to Frege's project aiming to settle the question of identity in mathematics by external logical means? It might seem that the notion of identity through change (transformation) invoked here remains completely informal. However in fact we have got a new formal concept of identity as *unity* of group of transformations. The group-theoretic notion of identity well complies with metaphysical intuition that identity through change involves a kind of repetition. Merging equivalent sets x, y, ... into same "set-substance" X indiscriminately we recover identity as a particular (unique) transformation. Prima facie it is unclear whether this group-theoretic identity has anything to do with the logical notion of identity Frege worries about. But in any event we have a well-defined identity concept here, which makes metaphysical intuitions behind it precise.

There are at least three immediate objection against the suggestion to take group-theoretic identity philosophically seriously.

(i) Logical (and metaphysical) notion of identity applies to wide domains of entities, so one can say which things in given domain are same and which are different. But group-theoretic identity relates to (identifies?) the only object, namely its group.

(ii) Group-theoretic identity does not allow us to form *propositions* like "A is identical to B". Generally group-theoretic identity is not a logical notion but a particular notion used in a particular branch of mathematics. So it has no general significance.

(iii) Moreover group-theoretic identity (like any other element of given group) is a particular mathematical object, which needs certain identity conditions of its own. These identity conditions essentially matter when one proves uniqueness of identity of given group. Hence one needs a logical notion of identity to cope with group-theoretic identity anyway.

Let us see how these objections can be met.

4. (De)Categorification

(i) This problem is fixed by generalizing the concept of group up to that of *category*. For this end one considers multiple objects with non-reversible transformations (morphisms) among them along with reversible ones (isomorphisms). The group-theoretic notion of identity is generalized to the effect that with each object A of the category is associated a particular (unique up to isomorphism) identity morphism 1_A such that $1_A f = f$ for any incoming morphism f and $g1_A = g$ for any outgoing morphism g. Since a category, generally speaking, comprises different objects, category-theoretic identity allows us not only to identify but also distinguish between objects.

(ii) Indeed prima facie Category theory says nothing about truth, proof, and inference, and these things seem to be indispensable in any branch of mathematics. However these and other logical notions are reconstructed by *internal* categorical means through category-theoretic construction of *topos*. I cannot discuss technical details here but would like to touch upon a more general question: whether or not logic in topos is *really* logic?

In my view the answer lies not in pure mathematics nor in philosophy of mathematics but in applied mathematics and pure metaphysics. For an apparent obstacle to answer the question in positive consists in the fact that prima facie topos-theoretic logic does not applies immediately to schoolish Socrates examples. If topos-theoretic logic would not apply but to mathematical matters it were not logic in a philosophically appealing sense. Thus the success of application of Category and Topos theory outside mathematics is crucial for taking these theories philosophically seriously. Pure metaphysics matters for a similar reason. Although Frege-Russell logic has had even less application in science than in mathematics, the metaphysical work made by these authors showed us how this logic (which had absorbed a good deal of what earlier would be counted as mathematics) applied to Socrates examples, and whatnot, and thus approved that it was *really* logic. The fact that Category-theoretic notions apparently better comply with the mathematical language of contemporary science than do Frege-Russell's logic and metaphysics is potentially a great advantage. However a metaphysical work is needed anyway to put category-theoretic notions down to the earth. Traditionally metaphysics was supposed to play the opposite role: to provide particular scientific disciplines with basic concepts and categories obtained through generalization upon pre-scientific reasoning about everything. But this link provided by metaphysics can and should be used also in the opposite direction to bring scientific thinking to common human affairs. Metaphysical work of Russell pursued exactly this goal: he tried to bring scientific way of reasoning (as he understood it) down to earth through a new logic rooted in the contemporary mathematics but applicable also to Socrates examples. Until this kind of work is done for Category theory its opponents can regard this theory as nothing but a particular mathematical theory and reject claims of its philosophical importance as wishful thinking. (One who detests metaphysics may do the job under the title of philosophical logic.)

(iii) To identify group- or more generally category-theoretic identity by internal means one repeats the trick and uses another group- (or category-)theoretic identity for it. Consider the above example of symmetric group . Let it be finite group S_N for simplicity. S_N "identifies" all (equivalent) sets of N elements by collapsing them into one. This collapse is not trivial because S_N has distinct elements, and in particular its identity 1. Now to identify S_N consider its own (auto-)transformations. This latter transformations also form a group called group $Aut(S_N)$ of automorphisms of S_N . Elements of $Aut(S_N)$ in their turn are identified through group of automorphisms $Aut^2(S_N)$) of $Aut(S_N)$, and so on. This looks like standard infinite regress (of the same kind as one involved into the Third Man paradox) however in the post-Cantorian epoque reductio ad infinitum should be hardly taken as reductio ad absurdum straightforwardly. Indeed the above construction continued unlimitedly (which might be called a *multigroup*) is surprisingly well-behaved. In the case N=2 all $Aut^n(S_N)$ are identities. In the case N>2 with a peculiar exception N=6 all $Aut^n(S_N)$ are isomorphic to S_N , so the infinite series gets stabilized immediately, and we have a sort of fix point here rather than regress, which takes us far[2]. (Isomorphisms between $Aut^n(S_N)$ form the same symmetric group S_N, of course.) In particular all identities $1^{(n)}$ map to (and only to) each other, so we can identify them all (again up to S_N) and talk about unique identity **1** of the whole symmetric multigroup.

In the case of other groups the construction is not so well-behaved though. In the case of an arbitrary category (when we have more than one object and non-reversible morphisms) the corresponding construction is called *multicategory* (*n-category* - partial construction at n-th step, and *ω-category* -full infinite construction). Apparently complexity of n-categories rises with n dramatically but as Baez&Dolan (1998) suggest the stabilization phenomenon, which we have observed in the case of symmetry multigroup, might be a fundamental property of multicategories or of a wide class of multicategories.

Construction of multicategories (from convenient mathematical objects) Baez&Dolan call *categorification* and describe it in the following words:

The basic philosophy is simple: *never mistake equivalence for equality* [italic of the author - AR]

and in a different place:

The recursive weakening of the notion of uniqueness, and therefore of the meaning of "the", is fundamental to categorification.

[2] See Kurosh (1955)

We see that the "philosophy" of categorification forbids exactly what does Frege's *abstraction* : taking equivalence for equality (or identity). Moreover the idea of categorification explicitly opposes Frege's project aiming to "strengthening the meaning of "the"" in mathematics (so it could be counted as full-fledged logical identity). The reason why Baez&Dolan talk about mistaking here is clear: taking equivalencies for equalities one loses group- and category structures, i.e. loses "information". Obviously such loss of information may cause errors if uncontrolled. However it is equally obvious that in many situations such loss of information (*decategorification*) is not only inevitable but non-trivial, productive, and desirable. Baez&Dolan discuss (informally) an example of decategorification, which would be particularly appealing for Frege: invention of natural numbers. As the story goes people constructed morphisms between sets of things long before they invented numbers. Numbers have been invented as the decategorification of category FinSet of finite sets (or rather its subcategory SmallFinSet not closed under sums and products) known from prehistoric times, likely as a result of mistake mentioned by Baez&Dolan. Apparently the notion of decategorification provides a better account of what is involved here than Frege's notion of abstraction. However further efforts are certainly needed to make the notion of decategorification logically clearer.

Literature:

Baez J.C., Dolan J. (1998), *Categorification* arXiv:math. QA 19802029 v1. (Internet document)

Frege, G. (1964) *The basic laws of Arithmetics*. Translated and edited, with an introduction by M. Furth; NY

Kurosh, A.G. (1955) *The Theory of Groups*, NY

Manin, Yu (2002) *Georg Cantor and His Heritage* : arXiv:math. AG/02009244 v1. (Internet document)