

Geometry, Logic and Axiomatic Method in the 21st Century. *Back to Euclid?*

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WARNING!

This paper is a piece of historical epistemology of mathematics, NOT of formal philosophy, i.e., not of philosophy using formal logical or other mathematical methods for its own purposes.

Brief History of Axiomatic Method

Hilbert

Euclid

Some reasons for dissatisfaction

Prospects

Categorical Logic and Axiomatic Topos theory

Homotopy Type theory and Univalent Foundations of
Mathematics

Conclusions

Hilbert 1899: *Grundlagen der Geometrie*

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C, \dots ; those of the second, we will call straight lines and designate them by the letters a, b, c, \dots ; and those of the third system, we will call planes and designate them by the Greek letters $\alpha, \beta, \gamma, \dots$. [...] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as “are situated”, “between”; “parallel”, “congruent”, “continuous”, etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry.

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- ▶ Non-Logical words (like *point*, *line*, *continuous*) have variable meaning which is assigned by an interpretation (model);

Hilbert circa 1900

[I]t is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps [...] and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. (Letter to Frege)

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- ▶ (3) Abstract “thought-things” (abstract structures ?)
- ▶ Question: What (1)-(3) share in common?

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- ▶ Propositional character.

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- ▶ Using Symbolic Logic;
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- ▶ Clarifying logical concepts by these mathematical means

Hilbert 1922

The axiomatic method is and remains the indispensable tool, appropriate to our minds, for all exact research in any field whatsoever: it is logically incontestable and at the same time fruitful. [...] To proceed axiomatically means in this sense nothing else than to think with consciousness.

Formal Method and Genetic Method

The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics. (Hilbert & Bernays 1934)

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [...], an axiomatic theory [in the narrow sense of “axiomatic”] refers to a fixed system of things (or several such systems) [i.e. to one or several models] [...] This is an idealizing assumption that properly augments [?] the assumptions formulated in the axioms.

V.A. Smirnov 1962

No theory can be developed without certain operations; one cannot proceed from what is given to anything else without performing certain operations. However in [Hilbert-style] axiomatic systems the only allowed operations are logical inferences, which are operations on propositions. In genetic theories operations on theoretical objects are equally allowed.

Remark

In a formal symbolic axiomatic setting logical inferences are construed as symbolic constructions. That means that in such a setting the genetic method is not abandoned altogether but limited to the domain of so-called *meta-mathematics*, which treats these symbolic constructions. There is a sense in which (so delimited) genetic method makes part of Hilbert-style formal axiomatic method (FAM) in its advanced symbolic version.

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- ▶ Postulates:
non-logical constructive rules

Axioms (Common Notions)

- A1. Things equal to the same thing are also equal to one another.
- A2. And if equal things are added to equal things then the wholes are equal.
- A3. And if equal things are subtracted from equal things then the remainders are equal.
- A4. And things coinciding with one another are equal to one another.
- A5. And the whole [is] greater than the part.

Axioms (continued)

Axioms apply to all mathematical objects universally (across the boundary between Geometry and Arithmetic). In this respect their role is similar to that of logical principles.

Aristotle uses the term “axiom” to refer to logical principles (e.g. to the perfect syllogism).

Postulates 1-3:

P1: to draw a straight-line from any point to any point.

P2: to produce a finite straight-line continuously in a straight-line.

P3: to draw a circle with any center and radius.

Postulates 1-3 (continued):

Postulates 1-3 are NOT propositions at all. They are not first truths. They are elementary (non-logical) *operations*.

Existential and modal interpretations of Postulates are possible but not necessary. They provide biased representations of Euclid's geometry.

Operational interpretation of Postulates

Postulates	input	output
P1	two points	straight segment
P2	straight segment	straight segment
P3	straight segment and its endpoint	circle

Shared Structure of Euclid's Problems and Theorems (after Proclus)

- ▶ *enunciation*
- ▶ *exposition*
- ▶ *specification*
- ▶ *construction*
- ▶ *proof* (in a narrow propositional sense)
- ▶ *conclusion*

Problem 1.1:

[*enunciation:*]

To construct an equilateral triangle on a given finite straight-line.

Problem 1.1 (continued):

[*exposition:*]

Let AB be the given finite straight-line.

specification:

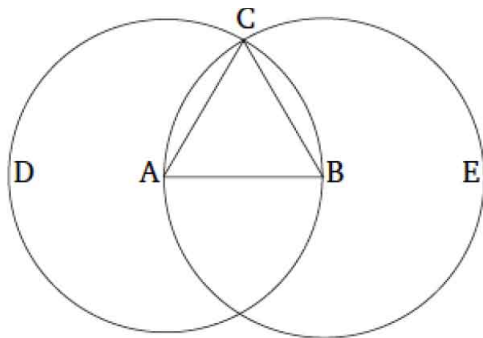
So it is required to construct an equilateral triangle on the straight-line AB .

Problem 1.1 (continued):

[*construction:*]

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C , where the circles cut one another, to the points A and B [Post. 1].

Problem 1.1 (continued):



Problem 1.1 (continued):

[*proof.*]

And since the point A is the center of the circle CDB , AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE , BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB . Thus, CA and CB are each equal to AB . But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB . Thus, the three (straight-lines) CA , AB , and BC are equal to one another.

Problem 1.1 (continued):

[*conclusion:*]

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB . (Which is) the very thing it was required to do.

Theorem 1.5:

[*enunciation:*]

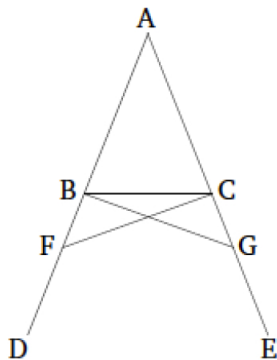
For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

Theorem 1.5 (continued):

[*exposition*]:

Let ABC be an isosceles triangle having the side AB equal to the side AC ; and let the straight lines BD and CE have been produced further in a straight line with AB and AC (respectively). [Post. 2].

Theorem 1.5 (continued):



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[*specification:*]

I say that the angle ABC is equal to ACB , and (angle) CBD to BCE .

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[*construction:*]

For let a point F be taken somewhere on BD , and let AG have been cut off from the greater AE , equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC , GB have been joined. [Post. 1]

Theorem 1.5 (continued):

[*proof.*]

In fact, since AF is equal to AG , and AB to AC , the two (straight lines) FA , AC are equal to the two (straight lines) GA , AB , respectively. They also encompass a common angle FAG . Thus, the base FC is equal to the base GB , and the triangle AFC will be equal to the triangle AGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG , and AFC to AGB . And since the whole of AF is equal to the whole of AG , within which AB is equal to AC , the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB .

Theorem 1.5 (continued):

[*proof* (continued):]

So the two (straight lines) BF , FC are equal to the two (straight lines) CG , GB respectively, and the angle BFC (is) equal to the angle CGB , while the base BC is common to them. Thus the triangle BFC will be equal to the triangle CGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus FBC is equal to GCB , and BCF to CBG . Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF , within which CBG is equal to BCF , the remainder ABC is thus equal to the remainder ACB [Ax. 3]. And they are at the base of triangle ABC . And FBC was also shown (to be) equal to GCB . And they are under the base.

Theorem 1.5 (continued):

[*conclusion:*]

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

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- ▶ Geometrical Construction makes proper part of Argument;
- ▶ Basic constructions (Postulates) are (non-propositional) principles of Euclid's geometry.
- ▶ Propositional deduction and geometrical *production* are intertwined. One requires the other (except trivial cases).

Kant on Euclid

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. [...] But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [...] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question.” (CPR, A 716 / B 744)

Remark

In a formal symbolic axiomatic setting we proceed similarly but delimit the domain of allowed constructions by purely syntactic (symbolic) constructions (and then think about their semantical interpretations). In Hilbert's view such a limitation provides an epistemic advantage by making all infinitary constructions unnecessary.

Friedman on Kant on Euclid

Euclidean geometry [..] is not to be compared with Hilbert's axiomatization [of Euclidean geometry], say, but rather with Frege's *Begriffsschrift*. It is not a substantive doctrine, but a form of rational representation: a form of rational argument and inference.

Pragmatic reasons to be dissatisfied with FAM

(1) FAM does not apply straightforwardly in the mainstream 20th c. maths. Hilbert's rigid distinction between mathematics and meta-mathematics in the context of the mainstream 20th c. maths. appears wholly artificial.

Example: Group theory is a model theory of the axiomatic group theory, i.e., the theory determined by the three group axioms. These axioms serve *only for defining* the concept of group. Most of theorems of groups theory (like Lagrange theorem) do not follow directly from these three axioms (just like the angle sum theorem of the Euclidean geometry does not follow directly from the definition of triangle).

Pragmatic reasons (continued)

(2) The impact of FAM on Set theory is unclear.

Example: The Independence of CH from ZF is well-established mathematical fact; the proof of this theorem (Gödel-Cohen) is not a formal axiomatic proof - notwithstanding the fact that this theorem treats a formal theory, namely ZF as its object (its subject-matter). This Independence result neither proves nor refutes CH. It does not allow to rule out CH as ill-posed either (after the example of Euclid's 5th Postulate). The full-scale relativism about mathematical statements is not consistent with the claim that the Independence of CH from ZF is well-established.

Pragmatic reasons (continued)

(3) The 20th c. showed no significant progress in the axiomatization of physics (Hilbert's 6th Problem). During this century FAM played no role at all in the mainstream research in physics and other natural sciences.

This reason, in my view, is the strongest one.

Epistemological reason

“Bond with Reality is cut” (Freudenthal).

Cf. : “Logical and mathematical concepts must no longer produce instruments for building a metaphysical “world of thought”: their proper function and their proper application is only within the empirical science” (Cassirer)

Epistemological reason (continued)

Hilbert-style formal mathematics anchors itself (in Cassirer's sense explained above) in the science and technology of *symbolic precessing* but NOT in the (fundamental) empirical science more broadly. This is epistemologically objectionable:

One must not forget that symbolic processing technologies (including today's IT) are ultimately made possible by our current knowledge of physics, NOT the other way round! This is notwithstanding the fact that all students of physics and all physicists use pens and computers.

Claim

Since 1960-ies we face a revival of the traditional genetic axiomatic method of theory-building in a wholly new mathematical setting. These recent developments suggest a substantial revision of the received (mainstream) philosophical views on foundations of mathematics, relationships between mathematics and logic, relationships between mathematics and natural science and some other. Some of these developments are philosophically motivated by the mathematical *intuitionism* and *constructivism* (Constructive Type theory).

Curry-Howard (a-historically): Simply typed lambda calculus

Variable: $\overline{\Gamma, x : T \vdash x : T}$

Product:
$$\frac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$$

$$\frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T} \quad \frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$$

Function:
$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \rightarrow T}$$

$$\frac{\Gamma \vdash t : U \rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$$

Curry-Howard (a-historically): Natural deduction

Identity: $\overline{\Gamma, A \vdash A}$ (Id)

Conjunction: $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$ (& - intro)

$\frac{\Gamma \vdash A \& B}{\Gamma \vdash A}$ (& - elim1); $\frac{\Gamma \vdash A \& B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ (\supset -intro)

$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ (\supset -elim aka *modus ponens*)

Curry-Howard Isomorphism

$$& \equiv \times$$

$$\supset \equiv \rightarrow$$

Brouwer-Heyting-Kolmogorov interpretation for Intuitionistic logic)

- ▶ proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B ;
- ▶ proof of $A \& B$ is a pair consisting of a proof of A and a proof of B

Carry-Howard and Euclid

In both cases constructions and logical deductions are intertwined (without being separated by the “meta-” barrier).

Historical Remark

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression “Curry-Howard isomorphism”, which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC)

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$$A \xrightarrow{f} B$$

- ▶ composition of morphisms: if $\text{Dom}(g) = \text{Cod}(f)$ then there exists unique composite $h = f \circ g$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

Mathematical Definition (continued)

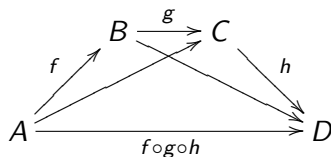
For each object A *identity morphism* 1_A with the following properties:

(i) for each incoming morphism $\xrightarrow{f} A$ we have $f \circ 1_A = f$

(ii) for each outgoing morphism $A \xrightarrow{g}$ we have $1_A \circ g = g$

Mathematical Definition (continued)

Composition of morphism is associative: $(f \circ g) \circ h = f \circ (g \circ h)$



the end of definition.

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- ▶ **Set**: sets and functions;
- ▶ **Top**: topological spaces and continuous transformations (mind the definition!) ;
- ▶ **Grp**: groups and group homomorphisms;
- ▶ a group is a category with a single object and all morphisms isomorphisms;
- ▶ a poset is a category having at most one morphism btw two given objects.

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- ▶ RELATION (on morphisms): $C(x, y; z)$ (composite)
- ▶ AXIOMS: (i) bookkeeping: if $C(x, y; z)$ then $Cod(x) = Dom(y)$ and $Dom(x) = Dom(z)$ and $Cod(y) = Cod(z)$; (ii) existence and uniqueness of composites, (iii) identity, (iv) associativity

CCC

is an (abstract) category with the terminal object, products and exponentials.

Examples: Sets, Boolean algebras

Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- ▶ Objects: types / propositions
- ▶ Morphisms: terms / proofs

Lawvere's philosophical motivation

- ▶ objective invariant structures vs. its subjective syntactical presentations (Structuralism)
- ▶ objective logic vs. subjective logic (Hegel)

History of CCC: Categorical Set theory

Idea (Lawvere early 1960ies): use functions instead of \in (back to von Neumann 1930ies'); take an abstract category and specify it “into” the (?) category of sets

ETCS (1)

There is a singleton 1: $\forall S \exists! S \rightarrow 1$

Every pair of sets A, B has a product:

$\forall T, f, g$ with $f: T \rightarrow A, g: T \rightarrow B, \exists! \langle f, g \rangle: T \rightarrow A \times B$

$$\begin{array}{ccccc}
 & & T & & \\
 & f \swarrow & \downarrow \langle f, g \rangle & \searrow g & \\
 A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B
 \end{array}$$

Every parallel pair of functions $f, g: A \rightarrow B$ has an equalizer:

$\forall T, h$ with $fh = gh \exists! u: T \rightarrow E$

$$\begin{array}{ccccc}
 & & T & & \\
 & & \searrow h & & \\
 u \downarrow & & & & \\
 E & \xrightarrow{e} & A & \xrightarrow[f]{g} & B
 \end{array}$$

ETCS (2)

There is a function set from each set A to each set B :

$$\forall C \text{ and } g: C \times A \rightarrow B, \exists! \hat{g}: C \rightarrow B^A$$

$$\begin{array}{ccc} C & & C \times A \xrightarrow{g} B \\ \hat{g} \downarrow & & \hat{g} \times 1_A \downarrow \nearrow e \\ B^A & & B^A \times A \end{array}$$

There is a truth value $true: 1 \rightarrow 2$:

$$\forall A \text{ and monic } S \rightarrowtail A, \exists! \chi_i \text{ making } S \text{ an equalizer}$$

$$S \rightarrowtail A \begin{array}{c} \xrightarrow{\chi_i} \\ \xrightarrow{true_A} \end{array} 2$$

ETCS (3)

There is a natural number triple $\mathbb{N}, 0, s$:

$$\forall T \text{ and } x: 1 \rightarrow T \text{ and } f: T \rightarrow T, \exists! u: \mathbb{N} \rightarrow T$$

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow x & \downarrow u & & \downarrow u \\ & & T & \xrightarrow{f} & T \end{array}$$

Extensionality: $\forall f \neq g: A \rightarrow B, \exists x: 1 \rightarrow A$ with $f(x) \neq g(x)$.

Non-triviality: $\exists \text{false}: 1 \rightarrow 2$ such that $\text{false} \neq \text{true}$.

Choice: \forall onto function $f: A \rightarrow B, \exists h: B \rightarrow A$ such that $fh = 1_A$.

History of CCC: Changing Axiomatic Method

The concept of CCC was discovered by Lawvere in 1969 (as a general setting for diagonal arguments) 5 years after he first axiomatized Set theory as a (first-order) ETCS in 1964. These 5 years mark Lawvere's shift from FAM to a new (or perhaps a "new-old") axiomatic method: instead of "using" the external (classical) FOL as logical foundation Lawvere now aims at building FOL internally as a part of his target axiomatic theory!

From this new viewpoint **Set** presents itself as a logical framework (rather than one "system of things" among others, which live in some pre-established logical framework). Cf. Bool's and Venn's semantic approach to logic: algebra of classes/propositions.

Calculus Ratiocinator versus Lingua Universalis (Heijenoort, Hintikka)

Higher-order generalization: Hyperdoctrines (Lawvere)

Quantifiers as adjoints (functors) to substitution (1969)

Locally Cartesian Closed Categories (LCCC, 1972)

Lawvere and Rosebrugh (2003) on the nature of Logic

The term “logic” has always had two meanings - a broader one and a narrower one:

- (1) All the general laws about the movement of human thinking should ultimately be made explicit so that thinking can be a reliable instrument, but
- (2) already Aristotle realized that one must start on that vast program with a more sharply defined subcase.

Lawvere and Rosebrugh (continued)

The achievements of this subprogram include the recognition of the necessity of making explicit

- (a) a limited universe of discourse, as well as
- (b) the correspondence assigning, to each adjective that is meaningful over a whole universe, the part of that universe where the adjective applies. This correspondence necessarily involves
- (c) an attendant homomorphic relation between connectives (like and and or) that apply to the adjectives and corresponding operations (like intersection and union) that apply to the parts “named” by the adjectives.

Lawvere and Rosebrugh (continued)

When thinking is temporarily limited to only one universe, the universe as such need not be mentioned; however, thinking actually involves relationships between several universes. [...] Each suitable passage from one universe of discourse to another induces (0) an operation of substitution in the inverse direction, applying to the adjectives meaningful over the second universe and yielding new adjectives meaningful over the first universe.

Lawvere and Rosebrugh (continued)

The same passage also induces two operations in the forward direction:

- (1) one operation corresponds to the idea of the direct image of a part but is called “existential quantification” as it applies to the adjectives that name the parts;
- (2) the other forward operation is called “universal quantification” on the adjectives and corresponds to a different geometrical operation on the parts of the first universe.

Lawvere and Rosebrugh (continued)

It is the study of the resulting algebra of parts of a universe of discourse and of these three transformations of parts between universes that we sometimes call “logic in the narrow sense”. Presentations of algebraic structures for the purpose of calculation are always needed, but it is a serious mistake to confuse the arbitrary formulations of such presentations with the objective structure itself or to arbitrarily enshrine one choice of presentation as the notion of logical theory, thereby obscuring even the existence of the invariant mathematical content.

Toposes (1970): Lawvere on logic and geometry

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck “topology” appears most naturally as a modal operator, of the nature “it is locally the case that”, the usual logical operators, such as \forall , \exists , \Rightarrow have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for “the” category \underline{S} of abstract sets to an arbitrary topos .

Lawvere on logic and geometry (continued)

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [...] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

(Elementary) topos is a category which

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(Elementary) topos is a category which

- ▶ has finite limits
- ▶ is CCC
- ▶ has a subobject classifier

New Features

Lawvere describes a topos as a logical framework by specifying its *internal logic* (by analogy with **Set**). *This* provides the wanted list of axioms for topos theory. In toposes a geometrical construction is a proper element of logical reasoning (through Curry-Howard Isomorphism).

MLTT (Martin-Löf 1980): key features

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- ▶ double interpretation of types: “sets” and propositions (imbedded Curry-Howard)

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- ▶ double interpretation of types: “sets” and propositions (imbedded Curry-Howard)
- ▶ double interpretation of terms: elements of sets and proofs of propositions
- ▶ higher orders: dependent types (types depending on terms of other types)
- ▶ MLTT is the internal language of LCCC (Seely 1983)

MLTT: two identities

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- ▶ Definitional identity of terms (of the same type) and of types:
 $x = y : A; A = B : \text{type}$ (substitutivity)

MLTT: two identities

- ▶ Definitional identity of terms (of the same type) and of types:
 $x = y : A$; $A = B : \text{type}$ (substitutivity)
- ▶ Propositional identity of terms x, y of (definitionally) the same type A :
 $Id_A(x, y) : \text{type}$;
 Remark: propositional identity is a (dependent) type on its own.

MLTT: Higher Identity Types

- ▶ $x', y' : Id_A(x, y)$
- ▶ $Id_{Id_A}(x', y') : type$
- ▶ and so on

Fundamental group

Fundamental group G_T^0 of a topological space T :

- ▶ a base point P ;
- ▶ loops through P (loops are circular paths $l : I \rightarrow T$);
- ▶ composition of the loops (up to homotopy only! - see below);
- ▶ identification of homotopic loops;
- ▶ independence of the choice of the base point.

Fundamental (1-) groupoid

G_T^1 :

- ▶ all points of T (no arbitrary choice);
- ▶ paths between the points (embeddings $s : I \rightarrow T$);
- ▶ composition of the *consecutive* paths (up to homotopy only! - see below);
- ▶ identification of homotopic paths;

Since not all paths are consecutive G_T^1 contains more information about T than G_T^0 !

Path Homotopy and Higher Homotopies

$s : I \rightarrow T, p : I \rightarrow T$ where $I = [0, 1]$: paths in T

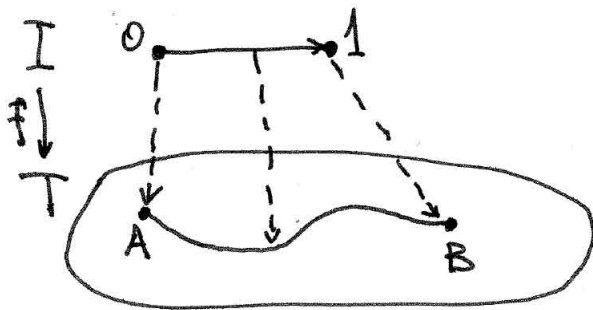
$h : I \times I \rightarrow T$: homotopy of paths s, t if $h(0 \times I) = s, h(1 \times I) = t$

$h^n : I \times I^{n-1} \rightarrow T$: n -homotopy of $n-1$ -homotopies h_0^{n-1}, h_1^{n-1} if

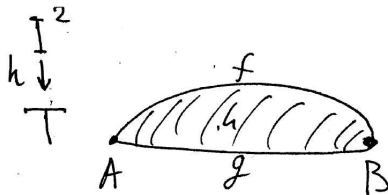
$h^n(0 \times I^{n-1}) = h_0^{n-1}, h^n(1 \times I^{n-1}) = h_1^{n-1};$

Remark: Paths are zero-homotopies

Path Homotopy and Higher Homotopies



Homotopy categorically and Categories homotopically



Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- ▶ all points of T (no arbitrary choice);
- ▶ paths between the points ;
- ▶ homotopies of paths
- ▶ homotopies of homotopies (2-homotopies)
- ▶ higher homotopies up to n -homotopies
- ▶ higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1} !

Composition of Paths

Concatenation of paths produces a map of the form $2I \rightarrow T$ but not of the form $I \rightarrow T$, i.e., not a path. We have the whole space of paths $I \rightarrow 2I$ to play with! But all those paths are homotopical. Similarly for higher homotopies (but beware that n -homotopies are composed in n different ways!)

On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure!

Grothendieck Conjecture:

G_T^ω contains all relevant information about T ; an omega-groupoid is a complete algebraic presentation of a topological space.

Homotopy Type theory

- ▶ Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). Extensionality one dimension up. (Streicher 1993).
- ▶ Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

h -levels

- ▶ (i) Given space is called *A contractible* (aka space of h -level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- ▶ (ii) We say that A is a space of h -level $n + 1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h -level n .

h -universe

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- ▶ Level 0: up to homotopy equivalence there is just one contractible space that we call “point” and denote pt ;

h -universe

- ▶ Level 0: up to homotopy equivalence there is just one contractible space that we call “point” and denote pt ;
- ▶ Level 1: up to homotopy equivalence there are two spaces here: the empty space \emptyset and the point pt . (For \emptyset condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call \emptyset, pt *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that h -level n corresponds to the logical level $n - 1$: the propositional logic (i.e., the propositional segment of our type theory) lives at h -level 1.

h -universe

h -universe

- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

h -universe

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- ▶ Level 3: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary flat) *groupoids* (with path spaces hom-sets).

h -universe

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- ▶ Level 3: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary flat) *groupoids* (with path spaces hom-sets).
- ▶ Level 4: 2-groupoids

h -universe

- ▶ ..
- ▶ Level $n+2$: n -groupoids
- ▶ ..
- ▶ ω -groupoids
- ▶ ω -groupoids ($\omega + 1 = \omega$)

How it works

Let $iscontr(A)$ and $isaprop(A)$ be formally constructed types “ A is contractible” and “ A is a proposition” (for formal definitions see Voevodsky:2011, p. 8). Then one formally deduces (= further constructs according to the same general rules) types $isaprop(iscontr(A))$ and $isaprop(isaprop(A))$, which are non-empty and thus “hold true” for each type A ; informally these latter types tell us that for all A “ A is contractible” is a proposition and “ A is a proposition” is again a proposition.

How it works

With the same technique one defines in this setting type $weq(A, B)$ of *weak equivalences* (i.e., homotopy equivalences) of given types A, B (as a type of maps $e : A \rightarrow B$ of appropriate sort) and formally proves its expected properties. These formal proves involve a *different* type $isweq(A, B)$ of h -level 2, which is a proposition saying that A, B are homotopy equivalent, i.e., that type $weq(A, B)$ is inhabited.)

Axiom of Univalence

Homotopically equivalent types are (propositionally) identical. This means that the universe *TYPE* of homotopy types is construed like a homotopy type (and also modeled by ω -groupoid).

Axiom of Univalence is the only axiom of Univalent Foundations on the top of MLTT.

Voevodsky on Univalent Foundations

The broad motivation behind univalent foundations is a desire to have a system in which mathematics can be formalized in a manner which is as natural as possible. Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. This problem becomes particularly pressing in attempting a computer formalization of mathematics; in the standard foundations, to write down in full even the most basic definitions - of isomorphism between sets, or of group structure on a set - requires many pages of symbols.

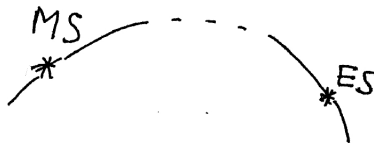
Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose syntax is tightly wedded to the intended semantical interpretation in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

Univalent Foundations: New Features

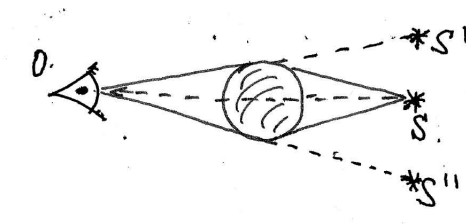
- ▶ Geometrical Intuition is vindicated;
- ▶ Formal Precision is saved;
- ▶ UA is the only non-logical (?) principle.

New Mathematical Principles for Natural Philosophy



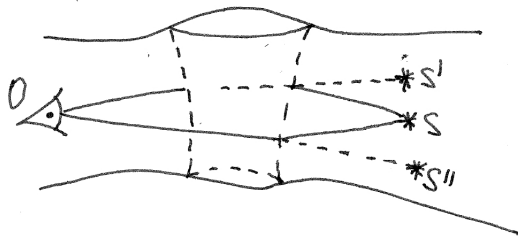
Identity through time

New Mathematical Principles for Natural Philosophy



Gravitational lensing

New Mathematical Principles for Natural Philosophy



Wormhole lensing

Conclusions:

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- ▶ The New Axiomatic Method of 21st century is the Good Old Genetic Axiomatic Method of Euclid, Newton and Clausius;

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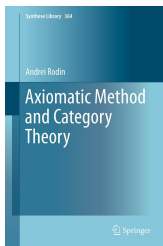
- ▶ The New Axiomatic Method of 21st century is the Good Old Genetic Axiomatic Method of Euclid, Newton and Clausius;
- ▶ There is no epistemic reason to privilege the symbolic intuition/construction over other sorts of mathematical intuition/construction.

Conclusions:

- ▶ The New Axiomatic Method of 21st century is the Good Old Genetic Axiomatic Method of Euclid, Newton and Clausius;
- ▶ There is no epistemic reason to privilege the symbolic intuition/construction over other sorts of mathematical intuition/construction.
- ▶ “The principle according to which our concepts should be sourced in intuitions means that they should be sourced in the mathematical physics. [...] Logical and mathematical concepts must no longer produce instruments for building a metaphysical “world of thought”: their proper function and their proper application is only within the empirical science” (Cassirer)



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A. Rodin, State University of Saint-Petersburg, Saint-Petersburg, Russian Federation
Axiomatic Method and Category Theory

Series: Synthese Library, Vol. 364

- ▶ Offers readers a coherent look at the past, present and anticipated future of the Axiomatic Method
- ▶ Provides a deep textual analysis of Euclid, Hilbert, and Lawvere that describes how their ideas are different and how their ideas progressed over time
- ▶ Presents a hypothetical New Axiomatic Method, which establishes closer relationships between mathematics and physics

This volume explores the many different meanings of the notion of the axiomatic method, offering an insightful historical and philosophical discussion about how these notions changed over the millennia.

The author, a well-known philosopher and historian of mathematics, first examines Euclid, who is considered the father of the axiomatic method, before moving onto Hilbert and Lawvere. He then presents a deep textual analysis of each writer and describes how their ideas are different and even how their ideas progressed over time. Next, the book explores category theory and details how it has revolutionized the notion of the axiomatic method. It considers the question of identity/equality in mathematics as well as examines the received theories of mathematical structuralism. In the end, Rodin presents a hypothetical

Contents

1	Introduction	1
Part I A Brief History of the Axiomatic Method		
2	Euclid: Doing and Showing	15
2.1	Demonstration and “Monstration”	16
2.2	Are Euclid’s Proofs Logical?	19
2.3	Instantiation, Objecthood and Objectivity	23
2.4	Proto-Logical Deduction and Geometrical Production	27
2.5	Euclid and Modern Mathematics	35
3	Hilbert: Making It Formal	39
3.1	<i>Foundations</i> of 1899: Logical Form and Mathematical Intuition ..	40
3.2	<i>Foundations</i> of 1899: Logicality and Logicism	47
3.3	Axiomatization of Logic: Logical Form Versus Symbolic Form ..	54
3.4	<i>Foundations</i> of 1927: Intuition Strikes Back	60
3.5	Symbolic Logic and Diagrammatic Logic	64
3.6	<i>Foundations</i> of 1934–1939: Doing Is Showing?	68
4	Formal Axiomatic Method and the Twentieth Century Mathematics ..	73
4.1	Set Theory	74
4.2	Bourbaki	78
4.3	Galilean Science and Set-Theoretic Foundations of Mathematics ..	87
4.4	Towards the New Axiomatic Method: Interpreting Logic	93
5	Lawvere: Pursuit of Objectivity	99
5.1	Elementary Theory of the Category of Sets	104
5.2	Category of Categories as a Foundation	105
5.3	Conceptual Theories and Their Presentations	110
5.4	Curry-Howard Correspondence and Cartesian Closed Categories ..	118
5.5	Hyperdoctrines	122
5.6	Functorial Semantics	125

5.7	Formal and Conceptual	126
5.8	Categorical Logic and Hegelian Dialectics	128
5.9	Toposes and Their Internal Logic	136

Part II Identity and Categorification

6	Identity in Classical and Constructive Mathematics	149
6.1	Paradoxes of Identity and Mathematical Doubles	149
6.2	Types and Tokens	152
6.3	Frege and Russell on the Identity of Natural Numbers	153
6.4	Plato	154
6.5	Definitions by Abstraction	156
6.6	Relative Identity	157
6.7	Internal Relations	158
6.8	Classes	160
6.9	Individuals	163
6.10	Extension and Intension	166
6.11	Identity in the Intuitionistic Type Theory	169
7	Identity Through Change, Category Theory and Homotopy Theory	175
7.1	Relations Versus Transformations	175
7.2	How to Think Circle	181
7.3	Categorification	183
7.4	Are Identity Morphisms Logical?	187
7.5	Fibred Categories	188
7.6	Higher Categories	190
7.7	Homotopies	193
7.8	Model Categories	199
7.9	Homotopy Type Theory	201
7.10	Univalent Foundations	204

Part III Subjective Intuitions and Objective Structures

8	How Mathematical Concepts Get Their Bodies	215
8.1	Changing Intuition	215
8.2	Form and Motion	217
8.3	Non-Euclidean Intuition	219
8.4	Lost Ideals	224
8.5	Are Intuitions Fundamental?	230
9	Categories Versus Structures	235
9.1	Structuralism, Mathematical	236
9.2	What Replaces What?	238
9.3	Erlangen Program and Axiomatic Method	242
9.4	Objective Structures	246

Contents	xi
9.5 Types and Categories of Structures	249
9.6 Invariance Versus Functionality	253
9.7 Are Categories Structures?	255
9.8 Objects Are Maps	257
10 New Axiomatic Method (Instead of Conclusion)	265
10.1 Unification	265
10.2 Concentration	267
10.3 Internal Logic as a Guide and as an Organizing Principle	268
Bibliography	273
Index	283

<http://arxiv.org/abs/1210.1478>

THANK YOU!