Genetic and Axiomatic Methods of Theory-Building from Hilbert to Voevodsky

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Abstract:

The so-called *genetic* way of building mathematical concepts and mathematical theories remains relevant in the current mathematical practice. On the contrary to Hilbert's 1900 view, which is still very popular in the logical and philosophical communities, the genetic method has not only heuristic and pedagogical but also a justificatory value. It admits a full formalisation in form of Univalent Foundations, which supports an automated formal proof-checking. This approach realises at the formal level what Hilbert and Bernays in 1934 refer to as the constructive method of theory-building. There are reasons to expect that this approach will be also more efficiently used in science beyond the pure mathematics than the standard Hilbert-style axiomatic method.

Genetic Method

Starting from the concept of the number 1, one usually imagines the further rational positive integers 2, 3, 4 ... as arising through the process of counting, and one develops their laws of calculation; then, by requiring that subtraction be universally applicable, one attains the negative numbers; next one defines fractions, say as a pair of numbers — so that every linear function possesses a zero; and finally one defines the real number as a cut or a fundamental sequence, thereby achieving the result that every entire rational indefinite (and indeed every continuous indefinite) function possesses a zero.

Genetic Method

We can call this method of introducing the concept of number the *genetic method*, because the most general concept of real number is *engendered* (erzeugt) by the successive *extension* of the simple concept of number.

Axiomatic Method

One proceeds essentially differently in the construction of geometry [in Hilbert 1899]. Here one customarily begins by assuming the existence of all the elements, i.e. one postulates at the outset three systems of things (namely, the points, lines, and planes) and then — essentially on the pattern of Euclid — brings these elements into relationship with one another by means of certain axioms, namely, the axioms of linking [Verknupfung], of ordering, of congruence, and of continuity.

Axiomatic Method

The necessary task then arises of showing the consistency and the completeness of these axioms, i.e. it must be proved that the application of the given axioms can never lead to contradictions, and, further, that the system of axioms is adequate to prove all geometrical propositions. We shall call this procedure of investigation the *axiomatic* method.

Remarks

- The introduction of reals as Dedekind cuts is *genetic* albeit not constructive in the usual sense of the term.
- The elementary geometry as presented in Euclid's Elements, Book 1, is genetic rather than axiomatic. In particular, Euclid's Postulates 1-3 are not propositions (axioms) but rules for constructing new geometrical objects from some given object. Hilbert recognises this fact in 1934 (see below).

Comparison of the two methods

My opinion is this: [d]espite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding (?) [Sicherung] of our knowledge the axiomatic method deserves the first rank.

Questions

- Does Hilbert have in mind the distinction between the context of discovery/learning and the context of justification? Probably yes.
- If so, what are his reasons to do this? Apparently his assumption about the foundational role of (informal) logic is such a reason. In 1934 he partly modifies this view.

Constructive Method

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Constructive Method

For axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory only as a *genus* of things, an axiomatic theory refers to a fixed system of things (or several such systems), and for all predicates of the propositions of the theory, this fixed system of things constitutes a *delimited domain of subjects*, about which hold propositions of the given theory. . . .

Constructive Method

We will call this sharpened form of axiomatics (where the subject matter is ignored and the existential form comes in) *formal axiomatics* for short.

On Euclid

Euclid's axiomatics was intended to be contentual and intuitive, and the intuitive meaning of the figures is not ignored in it. Furthermore, its axioms are not in existential form either: Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates.

Remark

Slightly anachronistically (by looking from Hilbert 1934 back to Hilbert 1900) Hilbert's misgivings about the genetic method can be explained as follows. Precise genetic procedures are available only for finite constructions such as finite symbolic constructions (the *real* mathematics). The rest of mathematics (called the *ideal* mathematics) requires a different grounding (where the genetic grounding remains at work but applies only to syntactic constructions and helps to establish certain facts about such constructions, e.g. formal consistency of a given theory (syntactically understood).

Remark

Hilbert's idea of dividing the whole of mathematics into the "real" and the "ideal" parts has a Kantian origin and is reductive in the sense that it doesn't grant to the new "ideal" mathematics an independent epistemic value. It doesn't stand against the 20th century mathematical practice. By Yuri Manin's word "good metamathematics is good mathematics rather than shackles on good mathematics".

The Hilbert-style formal axiomatic method proved effecient as a tool for meta-mathematical studies but not as a tool for representing mathematical knowledge. The *semantic* Bourbaki-style method of theory- building is a pragmatic compromise between the Hilbert-style axiomatic approach and the traditional contentful genetic mathematical reasoning. It allows for distinguishing between formalisable and non-formalisable proofs but does not support an efficient formalisation of mathematical proofs.

Research on Genetic Method in the 20th century

Cavailles 1938, Piaget 1956, Smirnov 1962

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$
  MLTT is a Gentzen-style rather than Hilbert-style formal system!

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

Sets and Propositions Are the Same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . .] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1983)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is <u>not</u> a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Homotopical interpretation of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- $\dots$

Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ *connected by a path with each point* $x : A$ *in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).*

Homotopy Levels

Definition

We say that S *is a space of h-level* $n+1$ *if for all its points* x, y *path spaces* $x =_S y$ *are of h-level* n*.*

Cummulative Hierarchy of Homotopy Types

- -2-type: single point *pt*;
- -1-type: the empty space $\emptyset$ and the point *pt*: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types ! An internal criterion of Logicality: Logic lives on *h*-level (-1)

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

How Logic relates to Higher (aka extra-logical) Types?

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Conclusions

Theories are essentially characterised by their proper *methods* of obtaining evidences that justify their statements. Such methods often have an heuristic power and help education but they also make part of the justificatory structure of a given theory.

Conclusions

Methods of justification of theoretical statements include *logical* methods (of obtaining evidences via logical inferences from given assumptions) and extra-logical constructive methods (or, as in case of HoTT/UF — extralogical applications of certain formal methods). Examples of such extra-logical methods are e.g. geometrical or algebraic methods in mathematics and experimental / observational methods in natural sciences. An extra-logical theoretical method has a logical impact once it helps the epistemic agent to obtain an evidence for a theoretical statement.

Conclusions

The idea of reduction of all theoretical methods to logical methods (in the above narrow sense), which is a part of Hilbert's approach, is in odds with the current mathematical and scientific practices and is not justified epistemologically.

Kiitos! Спасибо!