What is a Formal System?

The Idea of Geometrical Characteristics from Leibniz to Voevodsky

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Leibniz
Grassmann(s)
Peano
Russell
Hilbert
Tarski
Lawvere
Martin-Löf&Voevodsky
Conclusion

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Characteristica Universalis 1676

I'm thinking about a new language or a rational system of writing,
which could serve for communications between different peoples.
With such an instrument we could solve metaphysical and moral
problems just like geometrical problems. Any disagreement between
them will reduce to an error of calculation. Philosophers like
mathematicians could sit down and say: let's calculate.

Leibniz 1679: to Huygens

I believe that we must have still another properly linear geometrical analysis, which directly expresses *situm* as algebra expresses *magnitudem*.

Characteristica Geometrica 1679

Since the [traditional] letter notation of points in figures reflect geometrical properties of these figures, I wondered if any figure could be wholly represented by symbolic means in such a way that anll geometrical problems could be solved by manipulating with symbols. This cannot be done with Algebra alone since an algebraic solution is always supported with a geometrical proof.

Leibniz 1679: to Huygens

This new characteristic ... will not fail to give at the same time the solution, construction, and geometrical demonstration, the whole in a natural manner and by an analysis.

Hermann Grassmann (1847) Geometrische Analyse, geknüpft an die von Leibniz erfundene Geometrische Charakteristik.

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Die Formenlehre oder Mathematik. Von Robert Grassmann. Stettin,
1872.

Ersters Buch: Die Grösenlehre

Zweites Buch: Die Begriffslehre oder Logik

Drittes Buch: Die Bindelehre oder Combinationslehre

Viertes Buch: Die Zahlenlehre oder Arithmetik

Fünftes Buch: Die Ausenlehre oder Ausdehnungslehre.

Die Begriffslehre oder Logik

In order to ground the science of concepts or Logic, we should proceed formally and represent all proofs by equations, which are transformed according to rules provided by the science of Magnitude. Only this method of proof presupposes no logic and no grammar; this is the only method making thought rigorous. [. . .] Logic constitutes the second branch of the science of Form aka Mathematics, so it refers to definitions and rules of science of Magnitude.

Peano (1888)

Calcolo geometrico secondo l'Ausdehnungslehre di Hermann
Grassmann, preceduto delle operazioni della logic deductive

Geometric Formations:

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- numbers (the 0th species)

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- numbers (the 0th species)
- linear combinations of points (the 1st species)

Geometric Formations:

- ▶ numbers (the 0th species)
- ▶ linear combinations of points (the 1st species)
- ▶ linear combinations of lines (the 2nd species)

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- numbers (the 0th species)
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- linear combinations of lines (the 2nd species)
- linear combinations of triangles (the 3d species)

Geometric Formations:

- numbers (the 0th species)
- linear combinations of points (the 1st species)
- linear combinations of lines (the 2nd species)
- linear combinations of triangles (the 3d species)
- linear combinations of tetrahedra (the 4th species)

Operations on Formations:

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Operations on Formations:

- summing formations of the same species,

- multiplication of formations by number,

- progressive product for formations of species n, m given that $m + n < 4$)

Propositions as formations of (-1)th species?

propositional univalence

1. The *equivalence* of two entities **a** and **b** of the system is defined, that is, a proposition, indicated by **a** = **b**, is defined, which expresses a condition between two entities of the system, satisfied by certain pairs of entities, and not by others, and which satisfies the logical equations:

$$(\mathbf{a} = \mathbf{b}) = (\mathbf{b} = \mathbf{a}), \quad (\mathbf{a} = \mathbf{b}) \cap (\mathbf{b} = \mathbf{c}) < (\mathbf{a} = \mathbf{c}).$$

The geometric calculus is preceded by an introduction that treats of the operations of deductive logic; they present a great analogy with those of algebra and of geometric calculus. Deductive logic, which forms part of the science of mathematics, has not previously advanced very far, although it was a subject of study by Leibniz, Hamilton, Cayley, Boole, H. and R. Grassmann, Schroder, etc. The few questions treated in this introduction already constitute an organic whole, which may serve in much research. Many of the notations introduced are adopted in the geometric calculus.

Are classes "Formations of the zeroth species?

Russell: Principles of Mathematics 1903

[A]ll pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles.

Hilbert 1899 (1)

Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space.

Hilbert 1899 (2)

The following investigation is a new attempt to choose for geometry a simple and complete set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.

Hilbert 1827 (1)

No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction.

Hilbert 1827 (2)

This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert&Bernays 1934: Metamathematics

"Mathematics of concrete signs" belongs <u>not</u> to the theory expressed with these signs but to a separate theory.

Tarski 1959

[E]lementary geometry is ... a theory with standard formalization
. . . . It is formalized within elementary logic, i.e., first-order
predicate calculus. All the variables $x, y, z, \ldots$ occurring in this
theory are assumed to range over elements of a fixed set ; the
elements are referred to as points, and the set as the space. The
logical constants of the theory are [follows the usual list]. As
non-logical constants ... we pick two ... : the ternary predicate β
used to denote the betweenness relation and the quaternary
predicate δ used to denote the equidistance relation

"Standard Formalization"

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▶ Take a logical calculus L and fix its logical semantics

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"Standard Formalization"

- ▶ Take a logical calculus L and fix its logical semantics
- ▶ Take informal source theory G and use L for formulating non-logical axioms making sense (informally!) in G. An axiom system so obtained is a tentative formalization F of G.

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"Standard Formalization"

- ▶ Take a logical calculus L and fix its logical semantics
- ▶ Take informal source theory G and use L for formulating non-logical axioms making sense (informally!) in G. An axiom system so obtained is a tentative formalization F of G.
- ▶ In order to evaluate F against G check that theorems formally proved in F also make sense in G. As far as it works out F counts as an adequate formalization of G.

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Standard Formal System

= an axiom system obtained through a standard formalization. Such a FS is *propositional* in the sense that under the intended interpretation its WFFs are propositions. Since the underlying logic is fixed this feature is rigid; it cannot be altered through a different interpretation of non-logical symbols alone.

Lawvere 1970: internal geometrical logic

[A] Grothendieck "topology" appears most naturally as a modal
operator, of the nature "it is locally the case that", the usual logical
operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to
families of geometrical objects rather than to propositional
functions. [..] [I]n a sense logic is a special case of geometry.

Homotopy type theory: Univalent foundations of mathematics 2013

[W]hen types are viewed as propositions, they can contain more information than mere truth or falsity, and all "logical" constructions on them must respect this additional information. This suggests that we could obtain a more conventional logic by restricting attention to types that do not contain any more information than a truth value, and only regarding these as logical propositions. Such a type A will be "true" if it is inhabited, and "false" if its inhabitation yields a contradiction.

MLTT: two identities

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▶ Definitional identity of terms (of the same type) and of types:
$x = y : A$; $A = B :$ *type* (substitutivity)

MLTT: two identities

- Definitional identity of terms (of the same type) and of types:
  $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

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MLTT: extensional versus intensional

- ▶ Extensionality: Propositional identity implies definitional identity (ex. LCCC)

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MLTT: extensional versus intensional

- ▶ Extensionality: Propositional identity implies definitional identity (ex. LCCC)
- ▶ First intensional (albeit 1-extensional) model: Hofmann & Streicher 1994:
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

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Hofmann & Streicher groupoid model

$\vdash A : type$ - groupoid A

$\vdash x : A)$ - object x of groupoid A

$Id_A(x, y) : type$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
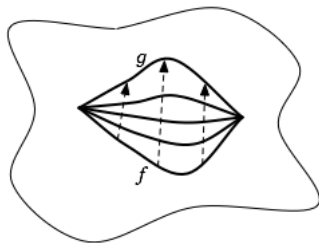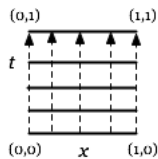
(no reason to be trivial unless $x = y$!)

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

HoTT: the idea

Types in MLTT can be modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in category theory. (Voevodsky circa 2008).

Path Homotopy

Homotopy model of MLTT

- In the groupoid model of MLTT groupoids are *fundamental groupoids* (i.e., groupoids of paths) of topological spaces .
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

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Definition:
Type is called a *mere proposition* if for all $x, y : A$ we have $x = y$

(homotopically: if A is *contractible*).

Carry-Howard explained

Every type is "reduced to proposition" through *truncation* of its higher-order structure, i.e., through identification of its terms.

As a formal system HoTT is <u>not</u> "merely propositional"! Thus it does not qualify as a standard formal system in the above sense.

Open Problem

Logical and philosophical discussion of the 20th century has been focused mostly - albeit not exclusively - on standard formal systems. This is why, in particular, the issue of *consistency* in its standard sense was so important. What of this discussion applies to non-standard non-propositional systems of Leibniz-Grassmann-Peano-Martin-Löf-Voevodsky and what remains to be done?