Rules versus Axioms: a Constructive View of Theories.

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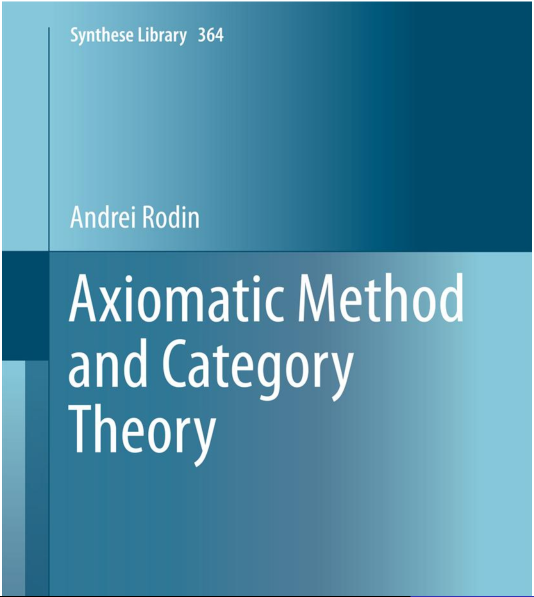
Project

Reinvention of the Axiomatic Method

Motivations:

- mathematics: the received methods of formalising mathematical theories don't support an effective formal proof-checking;
- novel axiomatic architectures emerging in the recent mathematical practice (Topos theory, Univalent Foundations);
- Hilbert's 6th Problem: little progress during more than a century;
- challenges of the current research in the program-based Knowledge Representation.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Axiomatic Method and Category Theory

Idea:

To use the Gentzen-style rule-based architecture for formal systems instead of the familiar Hilbert-style axiom-based architecture for representing scientific theories and other kinds of knowledge.

To make formal rules theory- and subject-specific and thus informative in this sense. Such rules may not qualify as *logical* under one's favourite conception of logicality. This is a very little explored dimension of the "axiomatic freedom" (that Hilbert himself to the best of my knowledge didn't consider seriously).

Perceived Advantages:

- facilitates computational implementations;
- allows for representing various *methods* (knowledge-how) including theoretical and empirical methods of verification/justification of statements (while methods of discovery can be arguable left out methods of justification cannot);
- combines representations of knowledge-that and of knowledge-how into a single formal framework;
- supports thought-experimentation and the experimental design.

Idea (more specific):

Use HoTT and Univalent Foundations as a paradigm for thinking about & representing & building theories in mathematics & science & technology.

Why HoTT? Because it provides a novel (unintended) semantics for MLTT that distinguishes between propositional and non-propositional (higher) types. This feature

- supports the representation of extra-logical methods and operations in theories such as methods of conducting physical experiments;
- at the same time it makes explicit the logical relevance of such extra-logical operations as verifiers of corresponding sentences.

Constructive View of Theories

A rule-based formal reconstruction of scientific theories gives rise to a view of theories, which for lack of a better name I call *constructive*. I propose this view as a replacement for the so-called *semantic view* first put forward by Patrick Suppes and some other people back in 1950-ies.

According to the semantic view a scientific theory should be identified not with some of its syntactic representations but with a class of models. The proposed constructive view agrees on this point with the semantic view. But the constructive view requires a notion of *modelling a rule*, which is absent from the standard Tarski-style Model theory.

Standard solution:

m is a model of rule R of form

$$\frac{A_1, \ldots, A_n}{B} \tag{1}$$

when the following holds: whenever $A_1^m, \ldots, A_n^m$ are true statements B^m is also true statement.

$$\frac{A_1^m, \ldots, A_n^m}{B^m} \tag{2}$$

Shortcomings of the Standard solution (hence an Open Problem):

- explains the rule away (at the semantic level) in terms of a meta-theoretical relation (viz., the logical consequence relation);
- assumes the standard notion of interpretation for formulas that involves a fixed distinction between logical and non-logical constants and variables (the homotopical interpretation of MLTT-formulas is not of this sort);
- applies only when interpreted formulas are statements but not tokens of non-propositional types, so it doesn't apply to extra-logical rules.

Example 1: Euclid

Axioms (Common Notions) and (at least some) Postulates in Euclid are *rules* but not sentences that admit truth-values, i.e., not axioms in the modern sense.

Many of Euclid's "Propositions" are Problems followed by Constructions while some other are Theorems followed by Proofs.

Problems and Theorems in Euclid share a common structure (an ancient prototype of Curry-Horward corespondence) and make part of a single deductive system, which is not adequately represented in standard modern axiomatic reconstructions of Euclid's geometry such as Hilbert's.

Euclid (continued)

Euclid's First Postulate reads verbatim as follows:

P1: *To draw a straight-line from any point to any point*

Observe that P1 is not a proposition but an elementary rule that validates a construction of straight line from a pair of distinct points.

Rendering Rules as Propositions (continued)

However in the modern (as well as in some older) versions of Euclid's theory it is usually replaced by one of the following propositions:

Given two (different) points it is always possible to produce a straight segment from one given point to the other given point.

Given two (different) points there exists a straight segment having these given points as its endpoint.

which are more apt for being formalized by standard logical means (even if the former requires a modal logic).

Rendering Rules as Propositions (continued)

I don't know about a logical theory that satisfactorily explains what is going on when "rules are translated into axioms" as above. In fact, these translation require a full rebuilding of the architecture of Euclid's theory (cf. Hilbert's 1899 version of Euclid). This is an evidence that such a propositional translation of rules is not logically innocent.

Example 2: Newton's *Principia*

Mathematical and experimental *methods* play a crucial role in the theoretical structure of the *Principia*. The title of the first Section of the first Book of Newton's *Principia* reads:

Of the Method of First and Last Ratios of Quantities

Example 3: Quantum Field Theory

comprises both mathematical methods (such as Renormalization methods) and very sophisticated experimental methods used, in particular in ATLAS and CMS experiments at CERN's LHC in 2012.

Do such experimental methods play a role in the *logical* structure of QFT? Yes, because they provide crucial *evidences* (proofs) for claims of this theory. Any reasonable logical analysis and any logical reconstruction of theories involves an analysis and reconstruction of its proofs.

Open Problem:

Is there a sense in which Hilbert-style and Gentzen-style formal representations of theories can be equivalent? If so, which classes of such representations are equivalent and which are not?

Propositional axiomatic theories

Definition

Propositional theory is a set T of formulae closed under application of the standard *modus ponens* (MP) (other rules are allowed but not required). Elements of T are called *theorems* of the given theory. The theory is called *axiomatic* when it comprises a distinguished subset $A \subset T$ of *axioms* such that all theorems of T are derivable from the axioms via applications of MP. The notion of derivation from a set Γ of hypotheses (denoted $\Gamma \vdash_T F$ or $\Gamma \vdash F$ when there is no risk of confusion) is standard.

Hilbertian theories

Definition

A propositional axiomatic theory is called *Hilbertian* when it comprises as theorems all formulae of the form $K_{A,B}$ and $S_{A,B,C}$ where

$$\begin{aligned} K_{A,B} &\doteq && A \rightarrow (B \rightarrow A) \\ S_{A,B,C} &\doteq && (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{aligned}$$

and has exactly one rule, namely MP.

Deduction Property

Definition

Theory T is said to have the *Deduction Property* if $\Gamma, F \vdash G$ entails $\Gamma \vdash F \to G$ for all Γ, F and G.

When a Hilbert-style and a Gentzen-style representation are deductively equivalent?

Theorem

An axiomatic propositional theory is Hilbertian if and only if it has the Deduction Property.

(Vladimir Krupski)

Lewis Regress

DP does not eliminate rules altogether but reduces their catalogue to *MP*

$A \vdash B$ if and only if $\vdash A \to B$
$A, A \to B \vdash B$ if and only if $A \vdash (A \to B) \to B$
$A, (A \to B) \to B \vdash B$ if and only if $A \vdash ((A \to B) \to B) \to B$
. . . .

Better leave $A \vdash B$ as it is? (cf. Lewis' example borrowed from Euclid).

Limitations:

- The first-order and higher-order cases are more involved;
- Deductive equivalences are syntactic and do not help one to translate between the two frameworks semantically, in particular, between a version of proof-theoretic semantics for rules and the Tarsky-style model-theoretic semantics for the logical consequence relation. The semantic difference is essential even if theories are deductively equivalent.
- Many system of interest (including Quantum Logics) do not have DP.

Martin-Löf 1983: Proof = Evidence

MLTT is a system of logic that is designed for managing proofs=evidences rather than only managing truth-values:

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

explanation of $t : T$ in Martin-Löf 1984

- t is an element of set T (Curry-Howard)
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T (Heyting)
- t is a method of solving the problem (doing the task) T (Kolmogorov)

explanation of $t : T$

If we take seriously , the idea that a proposition is defined by laying down how its canonical proofs are and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead, we simply identify them.

HoTT

"Types are Homotopy Types / Spaces."

This is obviously an *informal* interpretation, a mere "way of thinking of" and imagining elements of a formal syntactic system. Notice that unlike the case of Hilbert-style formal theories all (but not only non-logical) symbols and expressions of MLTT are interpreted here

One more item to the above list of informal interpretations? NOT just that.

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n + 1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-hierarchy

(0) : single point pt;

(1) : the empty space $\emptyset$ and the point pt : truth values aka *classical* or "mere" propositions

(2) : sets aka *intuitionistic*propositions aka theorems

(3) : (flat) groupoids

(4) : 2-groupoids

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(n) $n-2$-groupoids

- ...

(ω) ω-groupoids

HoTT semantics for $t : T$ for (1)-types

propositions and truth-values

HoTT semantics for $t : T$ for (2)-types

theorems and their proofs / sets and their elements

HoTT semantics for $t : T$ for higher -types

(also valid for lower types):

spaces and points, which support higher-order structures from elements of some other spaces (viz. map spaces);

objects are points;
constructions are points provided with additional higher-order structures: paths, surfaces (homotopies), etc.

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does nottake this structure into account. HoTT semantics does.

HoTT semantics (or the version thereof that I defend) does not license the idea that *every* type is a proposition and that sets and propositions are the same.

Instead it recovers the distinction between propositional (in the Classical sense) and non-propositional (higher) types and the distinction between logical inferences and extra-logical constructions. Logic belongs to the level (1) of the *h*-hierarchy. Set theory belongs to level 2, etc. Every extra-logical (= non-propositional) construction serves here as a proof / evidence for a proposition obtained by its (1)-truncation. Thus the schematic rules of MLTT are applied under this semantics both at the propositional level and at the higher *h*-levels.

Conclusion:

The new direction of the axiomatic freedom waits to be explored. It promises to provide a lot of useful applications in KR. Generally, Gentzen-style theories have no Hilbert-style counterparts. Informal linguistic translations between systems of rules and sets of axioms are not logically innocent and don't provide be themselves any formal equivalence relation.

Open Problem:

Translations between Hilbert- and Gentzen-style representation needs to be better understood at the formal semantic level.

thank you!

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