Category theory, Mathematical Structuralism and Mathematical Hermeneutics

Andrei RODIN

Paris 7 (Diderot)

rodin@ens.fr

"Mathematical Understanding", Paris, June 9-13, 2008

Plan of the Talk:

- Hermeneutics of Pythagorean Theorem: translation versus formalisation;

- Hilbertian Scheme and Categorical Theory-Building: Why Category Theory Does Not Support Mathematical Structuralism.

Reversed order of presentation

Hermeneutics of Pythagorean Theorem

(leaving proofs aside…)

(LM): Lang, S., Murrow, G., 1997, *Geometry*, Springer, p.95:
Let XYZ be a right triangle with legs of lengths x and y, and hypotenuse of length z. Then $x^2 + y^2 = z^2$.

(D): Doneddu, A., 1965, *Géométrie Euclidienne*, Plane, Paris, p.209, my translation:
Two non-zero vectors x and y are orthogonal if and only if $(y - x)^2 = y^2 + x^2$.

(E) Euclid, *Elements*, tr. by Th. Heath, Book 1, Proposition 47:
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

In which sense if any LM, D and E are different formulations (different versions) of the **same** theorem (provided we leave out in (D) its “if” part) ?

Let’s first *understand* them better!

(1) LM requires a theory of real numbers. The authors don’t provide such a theory. Instead they introduce the notion of distance through informally stated axioms of metric space and then mention that distances are numbers one reads off from a graduated ruler (?).

Side question:
Is a limited access to the Pythagorean Secret a **really good** pedagogical solution?

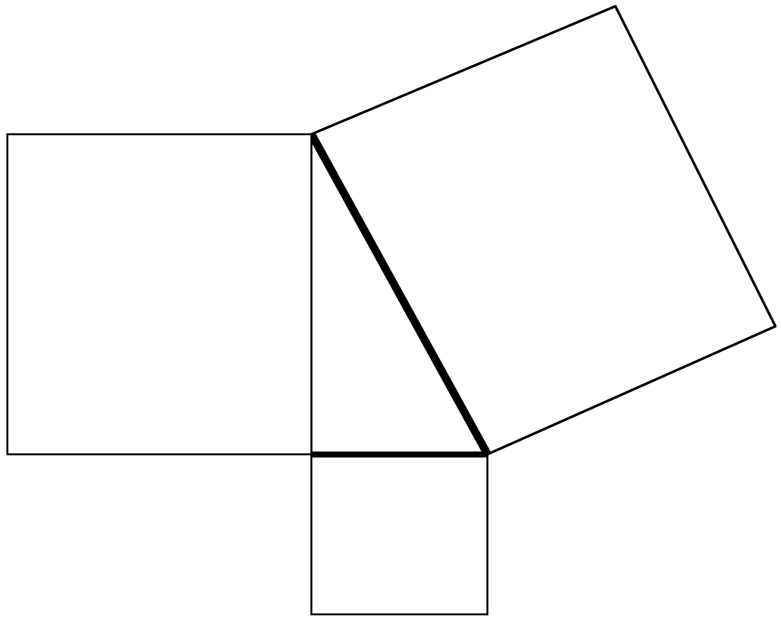
(2) D requires a theory of real numbers *and* basic Linear Algebra. The author meets the requirements. Notice that in the formula

$$(y - x)^2 = y^2 + x^2$$

- the minus sign on the left denotes the subtraction of *vectors* while the plus sign on the right denotes the sum of real numbers (so the two signs do *not* denote here reciprocal operations as usual)
- the square on both sides is understood in the sense of the scalar product of vectors.

The price of rigor turns to be high! No direct appeal to geometrical intuition.

(3) Beware that in I.47 Euclid doesn't speak about the equality of *areas* !



By "equality" Euclid means *equicomposability* or again more precisely….

Common Notions [=Axioms]:

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

3. And if equal things are subtracted from equal things then the remainders are equal.

4. And things coinciding with one another are equal to one another.

5. And the whole [is] greater than the part.

Notice (4): congruence is a special case.
Also mind Postulates.

(copy)
In which sense if any LM, (“only if” part of) D and E
are different formulations / versions of the **same** theorem?

Consider **very** different backgrounds (different foundations)
behind these statements. What is an appropriate background
for comparing them?

A further problem to be *only* mentioned her:

Mathematical truths like this one apparently survive through
changes of foundations? What makes it possible? How it works?
Does this provide a sense in which foundations don’t really matter?

Two possible strategies:

Usual Formalisation: extract a *form* / *structure* shared by **all** reasonable formulations of Pythagorean theorem. Do the same for the rest of Mathematics. This provides the wanted firm background, which can be used, in particular, for a better understanding of Mathematics of the past.

Problem:
What one gets through the usual formalisation is just another formulation / version F of the given theorem. (Arguably D qualifies as such.) What then justifies the claim that F *indeed* grasps all the essential features of other known formulations?

Usual answer: this is intuitively clear.

Usual argument supporting this answer: this is the best answer one can give. For F is rigor while other ("informal") formulations are not. One shouldn't require rigor talking about non-rigorous informal matters. If you really want to be rigorous work with F and forget about other formulations.

A critical reply:

The claim that F is (more) rigorous than "informal" formulations of the same theorem cannot be justified through the appeal to its formal character if the very notion of shared form (structure) is treated non-rigorously as suggested above.

It is historically naive and epistemically wrong to assume that today ’s standard "formal" foundations will survive forever. Revision of foundations is as much important for development of Mathematics as building upon assumed foundations. Revision of foundations doesn ’t cause giving up the rest. (Cf. Benabou on possible contradiction in ZF.) The architectural metaphor of Mathematics and Science is misleading!

The phenomenon of survival of mathematical knowledge through foundational changes should be taken seriously and treated rigorously.

A more precise proposal:
Step (1): study (construct) **translations** between ML, D and E and
Step (2): find appropriate identity conditions expressed in terms of these translations.

(1) What counts as a sound translation? How to justify a claim of the form "*A* translates into *B* by *t*" or diagrammatically

$$A \xrightarrow{\;t\;} B$$

when *A* and *B* are mathematical expression belonging to different theories (and, generally, sharing no conceptual core)?

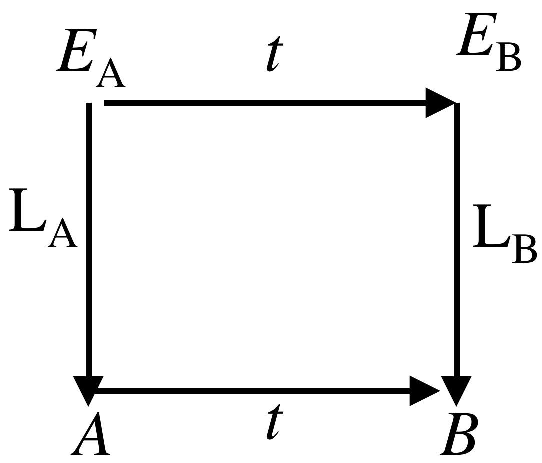
Hint: (i) internal and (ii) external **coherence**:

(i) *elements* of A translate into elements of B; translation **commutes** with *linking* the elements in A and B:

if $A = \mathrm{L_A} E_\mathrm{A}$, $B = \mathrm{L_B}\, E_\mathrm{B}$

then $t\, A = t\, \mathrm{L_A} E_\mathrm{A} = \mathrm{L_B}\, t\, E_\mathrm{A} = B$

Diagrammatically:



(ii) the same translation rules should apply outside *A* and *B*
(i.e. in wider domains belonging to correspondingtheories)

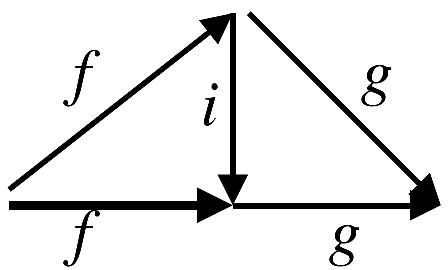
Ex.: E-->ML: magnitudes --> real numbers (measures)
Notice type forgetting: no backward elementwise translation
Very limited external domain.

Remark: sound translation *A*-->*B* needs not to be unique.

Ex.: projective duality: non-trivial translation of a given theory into itself.

(2) How to specify identity conditions through translation?

Consider the standard category-theoretic definition of identity:
i is identity iff $if = f$ for **all** incoming f **and** $gi = g$ for all outgoing g (provided the compositions exist).
diagrammatically:



Remarks:

A) one needs a notion of associative composition of translations (maps, morphisms);

B) there is no point to distinguish between an identity morphism and what it is identity of (an object);

C) this notion of identity is *strong*. One needs to specify a category an which it applies (notice the universal quantification).

D) This definition of identity is "contextual" (sensitive to neighbours)

Isomorphism in this context is **not** equivalent to identity but defined as follows:

f: *A*-->*B* is isomorphism iff there exist *g*: *B*-->*A* such that
(i) $gf = i_A$ and (ii) $fg = i_B$, where i_A and i_B are identities defined as above.

Remarks:
A) The mere existence of morphism going in the opposite direction is not sufficient (remind that morphisms *A*-->*B* are many);
none of (i) and (ii) is sufficient.
B) If the reverse exist it is unique.

(Rudimentary) Category theory suggests itself as the wanted background for the comparison. The result is context-dependent.
Another notion of identity morphism? (elsewhere)

Hilbertian Scheme and Categorical Theory-Building

"You say that my concepts, e.g. "point", "between", are not unequivocally fixed <...>. But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps <...> and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements.
For one merely has to apply a **univocal and reversible one-to-one transformation** and stipulate that the axioms for the transformed things be correspondingly similar. Indeed this is frequently applied, for example in the principle of duality, etc." (Hilbert's reply to Frege's critique of *Grundlagen* of 1899)

Structural setting of Hilbert's *Grundlagen* of 1899 popularised in the North America by Veblen and other *postulate theorists* and later elaborated by Tarski et al.:

Hilbertian scheme:
Formal theory + a bunch of its *isomorphic* models

Categoricity Problem (Veblen)**:**
Hilbertian scheme assumes that possible models of a given formal theory are **isomorphic**. But generally they are not. Hence the pursuit of categoricity. When it doesn't work (like in case of ZF) people often appeal to the notion of "standard" or "intended" model, which has no precise mathematical meaning. So intuitive considerations strike back! Hilbertian scheme doesn't work as it is supposed to.

General Anti-Structuralist Claim:
Hilbertian scheme doesn ’t work as it supposed to because

ALL MORPHISMS

but not only *isomorphisms* matter.

The pursuit of categoricity is unnecessary and misleading.

General Argument:

Hilbert has two very different notions of interpretation in mind. *First*, he thinks of interpretation of a given formal theory as an appropriate intuitive content, which can be associated with it. This is a philosophical, psychological and pedagogical issue but not a mathematical one. (Do different people imagine Euclidean circles differently?) *Second*, he thinks about a model M of a given formal theory T as a specific construction made within *another* theory T' (supplied by some working model M'). Hilbert's non-trivial mathematical examples are of this second kind. Think of arithmetical models of geometrical theories mentioned in Hilbert's *Grundlagen*.

Specific claims:

Claim 1:
There is no sufficient reason to treat both notions of interpretation on equal footing. This is a confusion of two very different things.

Argument:
I leave now the issue of intuition aside. But the *second* kind of interpretation can be better understood as a *translation* (*map*, morphism) between theories T and T' , i.e. interpretation of the theoretical content of T in terms of T'. This revised notion of interpretation (=translation) cannot be extended to the case of intuitive content (Hilbert's *first* kind of interpretation) because the intuitive content alone (whatever it might be)doesn't form anything like a theory.

Claim 2:
Hilbertian distinction between mathematics and meta-mathematics is not justified.

Argument:
The usual way to treat translation $T \dashrightarrow T'$ as interpretation in the *first* (intuitive) sense - to qualify deliberately T' as a **meta-**theory and on this ground to leave it out of mathematical consideration - in certain cases it leads to sheer epistemic absurdities (cf. Lobachevsky's "non-standard" model of Plane Euclidean geometry).

Claim 3:
Mathematically significant translations (maps, morphisms) between theories are generally non-reversible, i.e. not isomorphisms.

Argument:
Otherwise, according to Hilbertian criteria, they are auto-translations of a given theory into itself. Non-trivial reversible auto-translations exist (cf. Hilbert's example of Projective Duality) but are rare.
One shouldn’t generalise upon this Hilbert's example.

Remark:
Talking about arithmetical models of geometrical theories Hilbert, of course, didn't mean to identify Geometry with Arithmetic. But he thought he could "carve out" a specific arithmetical construction from its ambient theory and consider it (with appropriate arithmetical laws) as a self-standing embodiment of a geometrical theory.
This is not justified. The construction cannot survive outside its proper theoretical framework.

Claim 4:
Hilbertian scheme doesn't survive the replacement of isomorphisms by general morphisms.

Argument (crucial):
Given reversible map *A*<-->*B* one can think of *A*, *B* "up to isomorphism" and identify both *A*, *B* with a new "abstract" or "formal" object *C*. So differences between *A* and *B* can be dispensed with. This is possible because the existence of isomorphism is an equivalence relation, and *C* stands for a particular equivalence class by this relation. (Think about Frege's account of abstraction.) But the existence of general morphism *A*-->*B* is NOT an equivalence relation, so nothing similar applies in the general case. Given general morphism *A*-->*B* there is no sense in which the difference between *A* and *B* might not matter; there is no way to stipulate in this situation a new "formal" object C like in the special case of isomorphism (or in some similar way).

Remark:

Hilbertian Structuralist setting allows for a rigorous definition and treatment of the general notion of morphism. I mean the structuralist notion of morphism as a structure-preserving map. However this framework is based on a "preference" of isomorphisms to begin with. For the very notion of structure requires the kind of thinking exemplified by the above quote from Hilbert's letter to Frege.

Thinking about morphisms as structure-preserving is misleading.

Claim 5:
Set theory is a natural framework for applications of Hilbertian scheme (think of Tarski's semantics)

Argument (hint):
Any correspondence between two given elements of two given sets is (intuitively) reversible. In Set theory the notion of **non-ordered** pair is primitive (Pairing Axiom) but the notion of ordered pair is derived (construed in an artificial way). In this sense non-reversible correspondences between sets (i.e. functions) and maps between "structured sets" are accounted for in terms of elementary isos (i.e. pointwise).

Claim 5:
Category theory as a general theory of maps is a natural framework for the generalisation of Hilbertian scheme I'm pointing to.

Argument:
Presently we don't have any other proposal.

Remark:
Foundations of Category theory are not stabilized yet.This is a reason why Category theory is of philosophical interest.

Claim 6:
Categorical generalisation of Hilbertian scheme cannot be appropriately associated with a generalized version of Structuralism.

Argument:
Structures are specific categories but not the other way round (as it is often claimed).

Existing methods of categorical theory-building
(only to mention)

•Functorial semantics (Lawvere)
•Sketch theory (Ehresmann)

Common features:
•Categoricity in Veblen 's sense doesn 't make sense.
Instead one looks for
(i) "good" categorical properties of (categories of) models
(ii) specific models: generic, initial, universal, free
•Hilbertian distinction between formal theories and their models is blurred (Lawvere) or given up (Ehrehsmann).
•"Internalisation » of logic

Conclusions

- Category-theoretic approach to foundations of Mathematics supports a "hermeneutical" anti-foundationalist view on the history of the discipline.
- Category-theoretic approach doesn ’t support Mathematical Structuralism.

Thank You!