Category theory, Mathematical Structuralism and Mathematical Hermeneutics

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Plan of the Talk:

Reversed order of presentation

Support Mathematical Structuralism.
Building: Why Category Theory Does Not
Hilbertian Scheme and Categorical Theory

translation versus formalisation,

Hermeneutics of Pythagorean Theorem

Plan of the Talk:
Hermeneutics of Pythagorean Theorem.

The right angle is equal to the squares on the sides.

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

(E) Euclid, *Elements*, tr. by T.L. Heath, Book I, Proposition 47:

\[ a^2 + b^2 = c^2 \]


Two non-zero vectors \( x \) and \( y \) are orthogonal if and only if

\[ (y - x) \cdot x = 0 \]

(E) Euclid, *Elements*, tr. by T.L. Heath, Book I, Proposition 47:

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.


(Leaving proofs aside...)

Hermeneutics of Pythagorean Theorem.
In which sense, if any, are formulations (different versions) of the same theorem provided different in (D) and (E)? What we leave out in (D) is "if" part.

Let's first understand them better.

Is a limited access to the Pythagorean Secret a really good pedagogical solution?

Side question:

that distances are numbers one reads off from a graduated ruler (i.e.)

(1) LM requires a theory of real numbers. The authors don't provide such a theory. Instead, they introduce the notion of distance through informally stated axioms of metric space and then mention distances are numbers one reads off from a graduated ruler (i.e.)
The price of rigor turns to be high! No direct appeal to geometrical intuition.

Product of vectors.

The product of vectors is understood in the sense of the scalar product on both sides is understood in the sense of the scalar (so the two signs do not denote here reciprocal operations as usual) while the plus sign on the right denotes the sum of real numbers while the minus sign on the left denotes the subtraction of vectors.

\[ 2x + 2\lambda = \tau(x - \lambda) \]

The author meets the requirements. Notice that in the formula (2) D requires a theory of real numbers and basic Linear Algebra.
Beware that in I.47 Euclid doesn’t speak about the equality of areas!

By “equality” Euclid means equicomposability or again more precisely….
Common Notions (=Axioms):

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole is greater than the part.

Notice (4): congruence is a special case.

Also mind Postulates.
Does this provide a sense in which foundations don’t really matter?

Consider very different backgrounds (different foundations) behind these statements. What is an appropriate background for comparing them?

A further problem to be only mentioned here:

For comparing them?

In which sense if any LM, (“only if” part of) D and E are different formulations / versions of the same theorem?
Two possible strategies:

Usual Formalisation:

If you really want to be rigorous work with $F$ and forget about other formulations. You should require rigorous talking about non-rigorous informal matters. Give $F$. For it is rigorous while other (informal) formulations are not. One gives. $F$ is rigorous while other (informal) formulations are not. One gives.

Usual argument supporting this answer: this is the best answer one can give. For it is rigorous while other (informal) formulations are not.

Usual answer: this is intuitively clear.

Problem: What one gets through the usual formalisation is just another formulation of the given theorem. Arguably $D$ qualifies as such. What then justifies the claim that $F$ indeed grasps all the essential features of other known formulations?

Usual answer: this is the best answer one can give. For it is rigorous while other (informal) formulations are not. One should require rigorous talking about non-rigorous informal matters. If you really want to be rigorous work with $F$ and forget about other formulations.

Usual Formalisation: extract a form/structure shared by all reasonable formulations of Pythagorean theorem. Do the same for the Mathematics of the past. Which can be used, in particular, for a better understanding of the rest of Mathematics. This provides the wanted firm background, which can be used, in particular, for a better understanding of the Mathematics of the past.
A critical reply:
The claim that $F$ is (more) rigorous than informal formulations of the same theorem cannot be justified through the appeal to its formal character if the very notion of shared form (structure) is treated non-rigorously as suggested above. It is historically naive and epistemically wrong to assume that formal character at the very notion of shared form (structure) is the same theorem cannot be justified through the appeal to its revision of foundations as building upon assumed foundations. Revision of foundations is as much important for development of Mathematics as building upon assumed foundations. Revision of foundations doesn’t cause giving up the rest. (Cf. Benabou on possible contradiction in ZF.) Mathematics and Science is misleading! Mathematics and Science is misleading! The phenomenon of survival of mathematical knowledge through foundational changes should be taken seriously and treated rigorously. It is historically naive and epistemically wrong to assume that
A more precise proposal:

Step (1): study (construct) translations between ML, D and E and
Step (2): find appropriate identity conditions expressed in terms of

When A and B are mathematical expression belonging to different

\[ \text{theory}(\text{and, generally, sharing no conceptual core}) \]

What counts as a sound translation? How to justify a claim of the form "A translates into B by t" or diagrammatically

\[ A \xrightarrow{t} B \]

(1) These translations.

Step (2): find appropriate identity conditions expressed in terms of translations between ML, D and E and
Hint: (i) internal and (ii) external coherence

Diagrammatically:

\[ B = \varphi E \varphi B \]

then \( A \) linking the elements in \( A \) and \( B \):

(i) elements of \( A \) translate into elements of \( B \); translation commutes

\[ A = \varphi E \varphi A \]

\[ B = \varphi E \varphi B \]
The same translation rules should apply outside $A$ and $B$.

(i.e.) in wider domains belonging to corresponding theories.

(ii) Very limited external domain.

Notice type forgetting: no backward elementwise translation.

Remark: sound translation $A \rightarrow B$ needs not to be unique.

Ex.: projective duality: non-trivial translation of a given theory into itself.

Ex.: $\text{E} \rightarrow \text{ML}$: magnitudes $\rightarrow$ real numbers (measures).

(i.e.) in wider domains belonging to corresponding theories.
How to specify identity conditions through translation?

Consider the standard category-theoretic definition of identity:

\[ \text{is identity iff } \text{if } f = f \text{ for all incoming } f \text{ and } g = g \text{ for all outgoing } g \text{ (provided the compositions exist).} \]

Diagrammatically:

\[ \text{Diagrammatically: } \]

\[ \text{Consider the standard category-theoretic definition of identity:} \]

(2) How to specify identity conditions through translation?
Isomorphism in this context is not equivalent to identity but defined as follows:

\[ f : A \rightarrow B \] is isomorphism if there exist \( g : B \rightarrow A \) such that \( (i) \quad gf = i_A \) and \( (ii) \quad fg = i_B \), where \( i_A \) and \( i_B \) are identities defined as above.

Remarks:

A) The mere existence of morphism going in the opposite direction is not sufficient (remind that morphisms \( A \rightarrow B \) are many).

B) If the reverse exists, it is unique.

(Rudimentary) Category theory suggests itself as the wanted background for the comparison. The result is context-dependent.

Another notion of identity morphism? (elsewhere)
Building Hilbertian Scheme and Categorical Theory
Structural setting

of Hilbert's Grundlagen of 1899 popularised in the North America by Veblen and other postulate theorists and later elaborated by Tarski et al.: Hilbertian scheme:

Formal theory + bunch of its isomorphic models

Categoricity Problem (Veblen):

Hilbertian scheme: and later elaborated by Tarski et al.: in the North America by Veblen and other postulate theorists Structured setting of Hilbert's Grundlagen of 1899 popularised
The pursuit of categoricity is unnecessary and misleading.

but not only isomorphisms matter.

ALL MORPHISMS

Hilbertian scheme doesn't work as it supposed to because

General Anti-Structuralist Claim:
General Argument: Hilbert has two very different notions of interpretation in mind. First, he thinks of interpretation of a given formal theory as an appropriate intuitive content, which can be associated with it. This is a philosophical, psychological and pedagogical issue but not a mathematical one. Do different people imagine Euclidean circles differently?

Second, he thinks about a model $M$ of a given formal theory $T$ as a specific construction made within another formal theory $T'$ supplied by some working model $M'$. Hilbert’s non-trivial mathematical examples are of this second kind. Think of arithmetical models of geometrical theories mentioned in Hilbert’s *Grundlagen*. The first question he thinks about a model of a given formal theory $T$ as a specific construction made within another formal theory $T'$ is: are the two theories $T$ and $T'$ different? Second, he thinks about a model of a given formal theory as an appropriate intuitive content, which can be associated with it.
Claim 1: There is no sufficient reason to treat both notions of interpretation on equal footing. This is a confusion of two very different things.

Argument:
I leave now the issue of intuition aside. But the second kind of interpretation can be better understood as a translation (map) between theories and $T'$, i.e., interpretation of the theoretical content of $T$ in terms of $T'$. This revised notion of interpretation (translation) cannot be extended to the case of intuitive content (Hilbert's first kind of interpretation) because the intuitive content alone (whatever it might be) doesn't form anything like a theory.

Like a theory, content alone (whatever it might be) doesn't form anything.
Claim 2: Hilbertian distinction between mathematics and meta-mathematics is not justified.

Argument: The usual way to treat translation $T \rightarrow T'$ as interpretatively a meta-theory and on this ground to leave it out of mathematical consideration - in certain cases it leads to sheer epistemic absurdities (cf. Lobachevsky's "non-standard model of Plane Euklidean geometry.

The first (intuitive) sense - to qualify deliberately, $L \vdash \psi$ as a translation $T \rightarrow L\vdash \psi$ as interpretation in $L$.
Claim 3:
Mathematically significant translations (maps, morphisms) between theories are generally non-reversible, i.e., not isomorphisms.

Argument:
Otherwise, according to Hilbertian criteria, they are auto-translations of a given theory into itself. Non-trivial reversible auto-translations of a given theory into itself exist (cf. Hilbert’s example of Projective Duality) but are rare. Therefore, according to Hilbertian criteria, they are auto-translations of a given theory into itself. Non-trivial reversible auto-translations of a given theory into itself exist (cf. Hilbert’s example of Projective Duality) but are rare.

Remark:
One shouldn’t generalise upon this Hilbert’s example. Exist (cf. Hilbert’s example of Projective Duality) but are rare. Otherwise, according to Hilbertian criteria, they are auto-translations of a given theory into itself. Non-trivial reversible auto-translations of a given theory into itself exist (cf. Hilbert’s example of Projective Duality) but are rare.

Remark:
Talking about arithmetical models of geometrical theories Hilbert, of course, didn’t mean to identify Geometry with Arithmetic. But he thought he could “carve out” a specific arithmetical construction from its ambient theory and consider it (with appropriate arithmetical laws) as a self-standing embodiment of a geometrical theory.

This is not justified. The construction cannot survive outside its proper theoretical framework.
Claim 4:

Hilbertian scheme doesn't survive the replacement of isomorphisms by general morphisms.

Argument (crucial):

Given reversible map $A \rightarrow B$, one can think of $A$ and $B$ as identical. There is no sense in which the difference between $A$ and $B$ can be dispensed with. This is impossible because the existence of isomorphisms is an equivalence relation.

Given reversible map $A \rightarrow B$, one can think of $A$, $B$ with a new "abstract" or "formal" object $C$. So differences between $A$ and $B$ can be dispensed with. This is possible because the existence of isomorphisms is an equivalence relation, and $C$ stands for a particular equivalence class by this relation.

Think about Frege's account of abstraction. But the existence of general morphism $A \rightarrow B$ is NOT an equivalence relation, so nothing similar applies in the general case. Given general morphism $A \rightarrow B$, there is no sense in which the difference between $A$ and $B$ might not matter; there is no way to stipulate in this situation a new "formal" object $C$.
Remark:

Hilbertian Structuralist setting allows for a rigorous definition and treatment of the general notion of morphism. However, this framework is based on a "preference" of isomorphisms to begin with. Thinking about morphisms as structure-preserving is misleading.

Exemplified by the above quote from Hilbert's letter to Frege:

For the very notion of structure requires the kind of thinking exemplified by the above quote from Hilbert's letter to Frege.

Hilbertian Structuralist setting allows for a rigorous definition and treatment of the general notion of morphism. I mean the Structuralist

Remark:
Claim 5: Set theory is a natural framework for applications of Hilbert's scheme (think of Tarski's semantics).

Argument (hint): Any correspondence between two given elements of two given sets is (intuitively) reversible. In Set theory the notion of non-ordered pair is primitive (Pairing Axiom) but the notion of ordered pair is derived (constructed in an artificial way). In this sense non-reversible correspondences between sets (i.e. functions) and maps between structured sets (i.e. functions) and maps between "structured sets" are accounted for in terms of elementary pointwise.

Claim 5:
Claim 5: Category theory as a general theory of maps is a natural framework for the generalization of Hilbertian scheme I'm pointing to.

Argument:

Presently we don't have any other proposal.

Remark:

Foundations of Category theory are not stabilized yet. This is a reason why Category theory is of philosophical interest.

Claim 6: Categorical generalization of Hilbertian scheme cannot be appropriately associated with a generalized version of Structuralism.

Argument:

Structures are specific categories but not the other way round. As it is often claimed, Categories are specific categories but not the other way round.
Existing methods of categorical theory-building

- Functorial semantics (Lawvere)
- Sketch theory (Ehresmann)

Common Features:

- Categoricity in Veblen’s sense doesn’t make sense.
- Hilbertian distinction between formal theories and their models is blurred (Lawvere) or given up (Ehresmann).

Instead one looks for:

- Specific models: generic, initial, universal, free
- "Good" categorical properties of (categories of) models

(only to mention)

Existing methods of categorical theory-building
Conclusions

Category-theoretic approach doesn't support Mathematical Structuralism.

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- Category-theoretic approach does not support the discipline.
- Anti-foundationalist view on the history of Mathematics supports a "hermeneutical" view of Mathematics.
- Category-theoretic approach to Foundations.
Thank You!