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Identity and (De)Categorification

1. Identity, Equality, and Equivalence

Symbol "=" in expressions like 3=3 prima facie allows for two different interpretations: it may be read (i) as identity or (ii) as relation of equality between different *copies* (tokens, "doubles") of 3. Considering this ambiguity in his (1964) Frege opts for (i) and disproves usual way of thinking about numbers as existing in multiple copies. Geometrical examples show that such straightforward identification of mathematical equality with logical identity may not work in different contexts. For in geometry unlike arithmetic term *equality* may mean different things. There is a sense in which the "same figure" means the same shape and the same size, and there is another sense in which the "same figure" means only the same shape, and the "same shape" can be also specified differently. There is apparently no clear argument which would allow us to chose one of these senses of "same" as basic and eliminate others as the abuse of the language.

2. Definitions by Abstraction

Frege's method of definition by abstraction allows for reduction of multiple colloquial meanings of "same" in mathematics to universal logical identity concept through introduction of new abstract objects. This method is problematic from logical point of view (Frege himself finally rejects it) and in any event it does not provide what a mathematician is normally looking for. Frege's example of "direction" (not to be confused with orientation!) is hardly interesting mathematically; this notion might play at most an auxiliary role in geometry and can be easily dispensed with.

3. Relations versus Transformations

Replacement of given equivalence xEy by identity x=y proposed by Frege allows for a stronger interpretation than "abstraction". Namely, E can be interpreted as *reversible transformation,* which turns x to y and the other way round, and the identity = - as identity *through* this transformation. So we think here about given mathematical object as a *substance* capable for changing its states or positions. This substantialist reinterpretation of mathematical relations may look like an exercise in old-fashioned metaphysics but surprisingly it appears to be very fruitful from the mathematical point of view. Indeed the language of transformations is not formally equivalent to that of relations as one might expect but is richer. Given relation xRy there are, generally speaking many different transformations turning x into y, while xRy only says that there exists one. Moreover reversible transformations (of same object) form a certain structure called *group.* This fact remains completely hidden when one reduces transformations to relations. Among other things we get here a new (group-theoretic) identity concept (as unit of group).
Does this provide any viable alternative to Frege's project aiming to settle the question of identity in mathematics by external logical means? The following steps can be considered as a tentative realization of this project.

4. (De)Categorification

(i) The concept of group is generalized up to that of *category*. For this end one considers multiple objects with non-reversible transformations (morphisms) among them along with reversible ones (isomorphisms).

(ii) Although prima facie Category theory says nothing about truth, proof, and inference these and other logical notions are reconstructed by *internal* categorical means through category-theoretic construction of *topos*. I cannot discuss technical details here but would like to touch upon a more general question: whether or not logic in topos is *really* logic?
In my view the answer lies not in pure mathematics nor in philosophy of mathematics but in applied mathematics and pure metaphysics. The success of application of Category and Topos theory outside mathematics is crucial for taking these theories philosophically seriously. The fact that Category-theoretic notions apparently better comply with the mathematical language of contemporary science than do usual logic is potentially a great advantage. However a metaphysical work is needed anyway to put category-theoretic notions down to the earth. Traditionally metaphysics was supposed to play the opposite role: to provide particular scientific disciplines with basic concepts and categories obtained through generalization upon pre-scientific reasoning about everything. But this link provided by metaphysics can and should be used also in the opposite direction to bring scientific thinking to common human affairs. One who detests metaphysics may do the job under the title of philosophical logic.
(iii) Since group- or category-theoretic identities are particular mathematical objects they arguably need to be identified by external logical means anyway. To identify group- or category-theoretic identity by *internal* means one repeats the trick and uses another group- (or category-)theoretic identity for it. Consider finite symmetric group S_N for example. S_N "identifies" all (equivalent) sets of N elements by collapsing them into one. This collapse is not trivial because S_N has distinct elements, and in particular its identity 1. Now to identify S_N consider its own (auto-)transformations. This latter transformations also form a group called group $Aut(S_N)$ of automorphisms of S_N . Elements of $Aut(S_N)$ in their turn are identified through group of automorphisms $Aut^2(S_N)$) of $Aut(S_N)$, and so on. This looks like standard infinite regress but in the post-Cantorian epoque reductio ad infinitum should be hardly taken as reductio ad absurdum straightforwardly. Indeed the above construction continued unlimitedly (which might be called a *multigroup*) is surprisingly well-behaved. In the case N=2 all $Aut^n(S_N)$ are identities. In the case N>2 with a peculiar exception N=6 all $Aut^n(S_N)$ are isomorphic to S_N , so the infinite series gets stabilized immediately, and we have a sort of fix point here rather than regress, which takes us far (Kurosh 1955). (Isomorphisms between $Aut^n(S_N)$ form the same symmetric group S_N, of course.) In particular all identities $1^{(n)}$ map to (and only to) each other, so we can talk about identity **1** of the whole symmetric multigroup unique "up to itself".
In the case of an arbitrary category (when we have more than one object and non-reversible morphisms) the corresponding construction is called *multicategory*. Apparently complexity of n-categories rises with n dramatically but as Baez&Dolan (1998) suggest the stabilization phenomenon, which we have observed in the case of symmetry multigroup, might be a fundamental property of multicategories or of a wide class of multicategories.
Construction of multicategories (from convenient mathematical objects) Baez&Dolan call *categorification* and describe it in the following words:
The basic philosophy is simple: *never mistake equivalence for equality* [italic of the author - AR]

We see that the "philosophy" of categorification forbids exactly what does Frege's *abstraction* : taking equivalence for equality (or identity). The reason why Baez&Dolan talk about mistaking here is clear: taking equivalencies for equalities one loses group- and category structures, i.e. loses "information". Obviously such loss of information may cause errors if uncontrolled. However it is equally obvious that in many situations such loss of information (*decategorification*) is not only inevitable but non-trivial, productive, and desirable. Baez&Dolan discuss (informally) an example of decategorification, which would be particularly appealing for Frege: invention of natural numbers. As the story goes people constructed morphisms between sets of things long before they invented numbers. Numbers have been invented as the decategorification of category FinSet of finite sets (or rather its subcategory SmallFinSet not closed under sums and products) known from prehistoric times, likely as a result of mistake mentioned by Baez&Dolan. Apparently the notion of decategorification provides a better account of what is involved here than Frege's notion of abstraction. However further efforts are certainly needed to make the notion of decategorification logically clearer.

Literature:

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