VENUS HOMOTOPICALLY

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Abstract. The identity concept developed in the Homotopy Type theory (HoTT) supports an analysis of Frege's famous *Venus* example, which explains how empirical evidences justify judgements about identities. In the context of this analysis we consider the traditional distinction between the extension and the intension of concepts as it appears in HoTT, discuss an ontological significance of this distinction and, finally, provide a homotopical reconstruction of a basic kinematic scheme, which is used in the Classical Mechanics, and discuss its relevance in the Quantum Mechanics.

1. Introduction

According to Frege

> Identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur [6] (p. 254) [1]

In the second half of the 20th century this view was challenged by Peter Geach [10] who developed a theory of what he called the *relative identity*. Contrary to Frege, Geach holds that the identity concept allows for specifications, which depend on certain associated sortals [2].

Key words and phrases. identity, homotopy type theory, intension, kinematics.

[1] "Die Identitaet ist eine so bestimmt gegebene Beziehung, dass nicht abzusehen ist, wie bei ihr verschiedene Arten vorkommen können."

[2] Let a, b be parallel lines on Euclidean plane, in symbols $a//b$. Given that $//$ is an equivalence relation, Frege suggests to "take this relation as identity" (in symbols $a = b$) and thus obtain a new abstract object called *direction* ([7], p. 74e; for a more detailed reconstruction of Frege's abstraction see [23]). Geach's analysis of the same example is different: according to Geach $a = b$ reads "a and b are the same *as direction*" even if a and b are different *as lines*.

Geach's unorthodox view on identity has been never developed into an independent formal logical system and remain today rather marginal [2]. However the idea that, contrary to Frege's view, the identity concept can and should be diversified more recently reappeared in a different form in Martin-Löf's Constructive Type theory (MLTT) [14] and in the yet more recent geometrical interpretation of MLTT called Homotopy Type theory (HoTT) [16]. Unlike Geach's original proposal, which has hardly had any influence outside the philosophical logic, HoTT is a piece of new interesting mathematics and mathematical logic closely relevant to Computer Sciences.

The aim of this paper is to analyze some of Frege's ideas about identity in terms of the identity concept as it appears in MLTT and HoTT. In this way I hope to make the technical MLTT-HoTT identity concept more philosophically meaningful and more open to its possible applications in science.

The rest of the paper is organized as follows. In the next Section I present Frege's *Venus* example and overview its analysis by the author. In the following three Sections I introduce a basic fragment of MLTT and HoTT and discuss the difference between extensional and intensional versions of these theories. Then I present a reconstruction of Frege's *Venus* with HoTT and discuss in this context an ontological impact of the distinction between extensions and intensions. Finally, I extend my reconstruction of *Venus* to what I call the Basic Kinematic Scheme used in the Classical Mechanics and briefly discuss a possible relevance of this homotopical reconstruction in the Quantum Mechanics.

2. How identity statements are known?

Some identity statements are trivial and non-informative while some other are highly informative and in some cases very hard to prove. For example "2 = 2" (in words "two is two") is trivial, "2 is the only even prime number" is somewhat more informative but easy (since it follows immediately from the definitions of "even" and "prime"), while "2 is the biggest power n such that the equation $x^n + y^n = z^n$ has a solution in natural numbers" is both informative and highly non-trivial (it is a famous theorem conjectured by Pièrre Fermat in 1637 and proved by Andrew Wiles in 1994).

A non-mathematical example of the same kind is given by Frege in his classical *On Sense and Reference* [4] (English translation [5]). Frege considers three different names - *Venus*, *Morning Star* and *Evening Star* - which all refer to the same planet. Frege wonders how it is possible that while the identity statements

(1) *Venus* is *Venus* and

(2) *Venus* is *Morning Star*

are trivial (assuming that the latter statement expresses nothing but a linguistic convention) the statement

(3) *Evening Star* is *Morning Star*

is a non-obvious astronomical fact that needs an accurate justification, which involves both a specific theoretical background and suitable observational data [3].

Where does the difference between informative and non-informative identity statements come from? Frege does not provide a full answer to this question but does provide a theoretical framework for answering it. For this end he distinguishes between the *sense* and the *reference* of any given linguistic expression [4] . An identity statement of the form $a = b$ says that (and is tantamount of saying that) a and b have the same *reference*. The references of terms "a" and"b" fully determine the reference of expression "$a = b$", which according to Frege is the truth-value of the corresponding proposition. But on the top of that each of expressions "a", "b", and "$a = b$" has a certain *sense*. The sense of expression

[3]Instead of talking about trivial and non-trivial statements Frege uses here the Kantian distinction between synthetic and analytic judgements and also talk about the "cognitive value" of the corresponding thoughts. I shall not use Frege's original way of expressing these ideas in my presentation.

[4]Some writers who want to stress the originality of Frege's logical ideas leave Frege's German terms for *sense* and *reference* (Sinn und Bedeutung) without translation even if they write in English. I use standard English translations instead.

"$a = b$" is determined by the sense of "a", the sense of "b", and the way in which these two terms are combined. [5].

Whether an identity statement is informative or not depends on its sense (and hence on the sense of its constituents) but not on its reference. Thus there is no mystery in the fact that statements of the form $a = a$ are always trivial (assuming that both the sense of the reference of "a" is fixed), while statements of the form $a = b$ can be either trivial (when terms a, b have the same sense) or non-trivial (when terms a, b have different senses). In expressions (1) and (2) both terms have the same meaning (even if in (2) these terms differ linguistically) but in (3) the senses of two terms are different. This is why (1) and (2) are trivial but (3) is not.

Obviously this is not a complete explanation. Frege's system of symbolic logic aka *Begriffsschrift* [3] does not full justice to his own distinction between the sense and the reference of a linguistic expression [13]. It does not provide any specific logical rule for operating with senses of propositions. So Frege points to a problem but leaves it largely open. More recently a number of so-called *intensional* logical systems have been developed, some of them are motivated by the idea of formalizing certain aspects of Frege's *sense*. The distinction between extensions and intensions of linguistic expressions and logical terms is closely related to Frege's distinction between sense and reference [1]. It has a long history in logic and its philosophy and turns out to be instrumental in MLTT-HoTT, as we shall now see. In the next section I explain the technical meaning of this distinction in MLTT and then discuss its philosophical underpinning.

[5]This last thesis follows from a more general principle known as the *compositionality* of meaning. Frege is sometimes credited for the alleged invention of this principle but the true history is more complicated [15]

3. Extension and Intension

MLTT [14] comprises two different forms of identity concept [6]. These two forms of identity should look familiar to anyone who has at least a rudimentary experience in programming. It's one thing to assign to a certain symbol or symbolic expression its semantic value (which can be a number, a character, a string of characters and many other things) and it is quite a different thing to state that certain things are equal. (Hereafter I use words "equal" and "identical" interchangeably.) Only in the latter case one forms a *proposition*, which typically has precisely one of the two *Boolean* values: True and False. Outside the context of programming a similar distinction can be made between the *naming* or making some more elaborated linguistic convention, on the one hand, and making a *judgement* to the effect that certain things are equal, on the other hand. It's one thing to adopt and apply the convention according to which the goddess' name *Venus* is an alias for what is also known as the *Morning Star*, and it is, of course, quite a different thing to judge and state that two apparently different celestial objects such as the *Morning Star* and the *Evening Star* are, in fact, one and the same. In the latter case it is appropriate to ask for a proof. But such a demand would be obviously pointless in the former case.

The first kind of identity (one related to conventions) Martin-Löf calls *definitional* or *judgmental*; the second kind of identity he calls *propositional*. Following [16] I shall use sign "≡" for the definitional identity and the usual sign "=" for the propositional identity. Further, we should take *typing* into account. In MLTT both kinds of identity apply only to terms of the same type [7] Typing is reflected in the notation as follows:

$x, y : A$ is a judgment that states that terms x, y are of type A

Formula

[6]The original version of this theory involves *four* different kinds of identity ([14], page 59). I simplify the original account by deliberately confusing some syntactic and semantical aspects. Then we are left with the two forms of identity described below in the main text.

[7]I leave now aside how identity is applied in MLTT to *types* on the formal level. It is sufficient for my present purpose to talk about the "same type" and "different types" in MLTT informally.

(1) $x \equiv_A y$

stands for a judgement, which is tantamount to a convention (aka *definition*) according to which terms x, y of the same type A have the same meaning. Given (1) one says that x, y are *definitionally equal*.

The expression

(2) $x =_A y$

, in its turn, stands for a *proposition* saying that terms x, y of type A are equal. (2) is not a judgment but a new *type*. Under the intended proof-theoretic semantic of MLTT any term p of this type is thought of as a *proof* of the corresponding proposition; in the proof-theoretic jargon proofs are also called *witnesses* and sometimes *evidences*. So the following judgement

(3) (2) $p : x =_A y$

states that terms x, y are (propositionally) equal as this is evidenced by proof p.

Let us now see what kind of thing such a proof p can possibly be. In MLTT definitionally equal terms are interchangeable *salva veritate* as usual. Under the intended semantic of this theory this means that definitionally equal terms are interchangeable as proofs. This property of $\equiv$ and the reflexivity of $=$ justify the following rule

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv refl_x$ is built canonically ([16], p. 46). In words: the definitional identity (equality) implies the propositional identity (equality).

The converse rule is called the *equality reflection rule* or ER for short:

$$\frac{p : x =_A y}{x \equiv_A y}$$

In words: the propositional identity implies the definitional identity.

ER does not follow from other principles of MLTT but may be assumed as an independent principle. In this case one obtains a version of MLTT, which is called (definitionally) *extensional*. MLTT without ER is called *intensional*. It can be shown that in the extensional MLTT any (propositional) identity type $x =_A y$ is either empty or has a single term, namely $refl_x$, which is the canonical proof of this identity "by definition".

We see that ER makes the distinction between the definitional and the propositional identity purely formal and epistemologically insignificant. This feature of extensional MLTT can be viewed as a desirable conceptual simplification but it comes with a price. A significant part of this price concerns computational properties in MLTT and is important for applications of this theory in programming: while the intensional MLTT is decidable but the extensional MLTT is not. I shall not discuss this technical feature in this paper. Instead I shall argue that the intensional MLTT has also important epistemic advantages over its extensional cousin.

4. Fixing Identities or Leaving Them Evolving?

As we have seen in the extensional MLTT every identity is grounded in a definition. In order to apply this formal theory in reasoning one needs to fix in advance via definitions exact identity conditions for all objects involved in a given reasoning. This logical and epistemic requirement is known in the form of slogan "no entity without identity" due to Quine. It is interesting to notice that Quine himself does not accept this slogan without reservations. In Quine's view the slogan applies only in scientific reasoning and, moreover, only in the contemporary form of scientific reasoning. Bulk terms (aka mass terms) like "water", according to Quine, are remnants of an archaic logical scheme, which does not involve the individuation in its today's form. Quine further speculates that the contemporary "individuative, object-oriented conceptual scheme" can be replaced in a future by a different scheme, that will provide a "yet unimagined pattern beyond individuation" ([17], p. 24) [8]. In what follows I argue that the *intensional* MLTT along with HoTT provides such a pattern "beyond individuation" or at least a pattern of individuation beyond its usual extensional mode. But beforehand I would like to stress once again that the standard extensional mode of individuation is not sufficient for certain well-recognized and important scientific purposes. Frege's *Venus* example, if one takes it seriously, demonstrates this clearly. Fixing the identity of *Morning Star* and *Evening Star* and *Venus* via a definition is a prerequisite for applying a standard extensional logical scheme in any reasoning about this celestial object. This condition makes it impossible to support with such a scheme a reasoning, in which the identity of the *Morning Star* and the *Evening Star* is established on the basis of certain sufficient evidences.

Frege's example shows that "half-entites inaccessible to identity" ([17], p. 23) may look more familiar than Quine's colorful language suggests. In the *Venus* case we deal with a relatively innocent violation of "no entity without identity" requirement. We start with certain well-defined objects such as the *Morning Star* and the *Evening Star* but do not exclude the possibility that these objects can be eventually proved to be the same even if this fact does not follow from their definitions. Following Quine one may speculate about further deviations from the standard extensional scheme and attempt to conceive of a scheme, which does not use the definitional form identity at all. I shall not pursue

[8] Here is the full quote:

"[W]e may have in the bulk term a relic, half vestigial and half adapted, of a pre-individuative phase in the evolution of our conceptual scheme. And some day, correspondingly, something of our present individuative talk may in turn end up, half vestigial and half adapted, within a new and as yet unimagined pattern beyond individuation. Transition to some such radically new pattern could occur either through a conscious philosophical enterprise or by slow and unreasoned development along lines of least resistance. A combination of both factors is likeliest [. . .] " ([17], p. 24) this further project in this paper. Instead I shall show how the innocent-looking modification of the extensional scheme, which has been just explained, results into a remarkable diversification of the standard identity concept.

5. Higher Identity Types

Recall that the intensional version of MLTT has been introduced above via a negative characteristic: it is simply the MLTT *without* the reflexion rule ER.

The absence of ER allows for constructing further identity types as follows. Suppose we have a propositional identity type and a pair of terms of this type:

$x', y' : x =_A y$

Terms x', y' witness here the identity of terms x, y. It may happen that these two witnesses are in fact the same and that this fact is witnessed by two further terms x'', y'':

$x'', y'' : x' =_{x=_A y} y'$

This tower of higher identity types can be continued indefinitely. In the general case such a structure may have, of course, more than just two elements on each level.

Until the late 1990-ies the mathematical structure of this formal syntactic construction remained opaque. Since the *intentionality* in MLTT is a mere lack of the *extensionality* any model of the extensional MLTT also qualifies as a model of the intensional version of this theory. In 1994-1998 Hofmann and Streicher [11], [12] published the first non-extensional model of MLTT where the first-level identity types were modeled by abstract groupoids. This model allows the first-level identity types (i.e., types of the form $x =_A y$ where A is a type other than identity) to have multiple non-trivial terms (proves) but does not allow the same for all higher identity types. In other words, this model verifies the condition called "extensionality one dimension up". A deeper insight into the structure of higher identity types has been obtained around 2006 when Awodey and Voevodsky independently observed that the abstract groupoids of Hofman and Streicher's model can be thought of as *fundamental groupoids* (i.e., groupoids of all continuous *paths*) of topological spaces and be further extended to homotopy- and higher-homotopy groupoids of the same spaces, which model higher-order identity types of MLTT. Thus the Homotopy theory allows for building models of MLTT, which are "intensional all the way up". In such models the identity types of all levels are modeled uniformly. This discovery marked the emergence of a new theory known today under the name of Homotopy type theory and of a closely related foundational project called the Univalent Foundations of mathematics. For a systematic exposition of HoTT I refer the reader to [16] [9]

Beware that unlike Russell's type theories and their likes HoTT does not form its hierarchy of types by considering, first, classes of individuals, second, classes of such classes, and so on. The hierarchy of types in HoTT is of a geometric or, more precisely, homotopic nature. Sets are taken to be types of zero level. Terms of 0-types are points having non-trivial paths between them. Terms of 1-types are points allowing for non-trivial paths between them, but not allowing for non-trivial homotopies between these paths. Terms of 3-types allow for paths and non-trivial homotopies but not for non-trivial higher homotopies. And so on. [10]. A given n-type can be transformed into its underlying m-type with $m < n$ by forgetting (or, more precisely, by trivializing) its higher-order structure of all levels $> m$. Such an operation is called in HoTT *truncation*.

The logical significance and the possible epistemic function of higher identity types in MLTT are not yet well understood. The present work is an attempt of filling a part of this gap. In what follows consider only 0- and 1-types and leave a study of higher identity types for a future work.

[9] Since this area of research is rapidly developing, the 2013 book [16] does not include certain new results and developments. However it provides an systematic introduction, which is more than sufficient for my present purpose.

[10] Here I follow [16], see p. 99-100. On an alternative count the 0-type is a single point, 1-types are propositional types while sets are 2-types. The count adopted in [16] is more natural from a logical point view (given the current usual understanding of logic) while the latter count used by Voevodsky in his lectures is more natural from a geometric point of view.

6. Is Frege's *Venus* Example Linguistic?

Apparently Frege treats his *Venus* example as purely linguistic on equal footing with his other examples, which involve Alexander the Great, Columbus, Napoleon, Kepler dying in misery, Bucephalus and what not. Accordingly, the main result of his classical paper [4], [5], namely the distinction between the sense and the reference of a given linguistic expression, belongs primarily to the philosophy of language. Frege scholarship mostly follows Frege in this respect: a linguistic leaning aka *linguistic turn*became a brand mark of the influential *Analytic* branch of the 20th century and today's philosophy. It is quite remarkable, however, that when Frege first introduces and explains the *Venus* problem he does this not only in linguistic terms:

> The discovery that the rising Sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even today the identification of a small planet [i.e., an asteroid - *A.R.*] or a comet is not always a matter of course. ([5], p. 56)

The idea that a logical analysis of ordinary language can be helpful for solving problems of object identification in science in general and in astronomy in particular is based on Frege's strong assumption according to which the identity concept is the same in all these cases, so that "it is inconceivable that various kinds of it should occur" (see the full quote and the reference in the above Introduction). Without trying to challenge now this approach on the methodological level I shall provide here an alternative analysis of the same example, which takes its physical content and, even more importantly, its related mathematical form more seriously and applies some basic elements of HoTT introduced in the previous Section. As a matter of course this reconstruction is not intended to be a piece of mathematical physics. Nevertheless it provides a novel formal approach to traditional metaphysical issues concerning the identity through time and motion, which may be possibly helpful for dealing with identity-related problems of modern physics [8], [9].

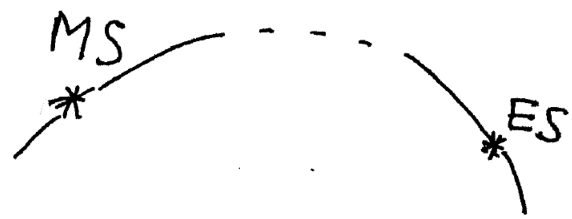

FIGURE 1. Morning Star and Evening Star are the same!

Frege's remark about the rising Sun quoted above applies both to the *Morning Star* (MS for short) and to the *Evening Star* (ES). These two putative objects are posited as invariants of certain sets of observations made in different places at different times by different people with different astronomical instruments and with the naked eye. However for the sake of the example I leave now this complex underlying structure aside and boldly assume that MS and ES are provided with appropriate *definitions*, which allow all observers to identify these objects unambiguously. How a *proof* of identity $MS = ES$ may look like in a proper realistic context? Classical Celestial Mechanics (CM), or more precisely a very basic fragment of CM that I shall call Basic Kinematic Scheme (BKS) and discuss in more detail in Section 8, provides a definite answer to this question. In order to prove that $MS = ES$ it is necessary and sufficient to present a *continuous path* aka trajectory p, which connects MS and ES and thereby shows that these "two" objects are in fact one and the same. The wanted trajectory p is itself a typical physical object: it is obviously theoretically-laden, it has a canonical mathematical representation, and it is accessible for observations which allow for empirical checks of its theoretically predicted properties. Providing such a proof p amounts to a combination of theoretical work and observation, which is typical in astronomy and any other mature science.

Since p has empirical contents and doesn't go through without such contents it can not be called formal. However the *form* of this proof is expressed within HoTT straightforwardly. As we shall briefly see, this form qualifies both as logical and geometrical. The fact that in

HoTT logical and geometrical forms go together, makes HoTT quite unlike other popular formal systems such as the Classical First-Order Logic (FOL), see my [18], ch. 7, 10 for a further discussion on this general issue. Remarkably, the geometrical form of p provided by HoTT (namely, a path) and the standard geometrical representation of the same object provided by CM and BKS (namely, a continuous curve) are nearly the same [11].

First, we need to specify a type (which under the homotopical interpretation is thought of as a space) where MS and ES belong. Since MS, ES and other celestial bodies are conceived in CM as point-like objects I call the corresponding type/space Pt and think of it as a collection of points:

$MS, ES : Pt$

Then we form a new type/space $MS =_{Pt} ES$, which is a space of continuous paths between MS and ES. Finally, we specify a particular path p in this space:

$p : MS =_{Pt} ES$

However little of HoTT's resources we use here, this reconstruction of Frege's example provides some useful lessons as we shall now see [12].

[11]In Homotopy theory a *path* is not simply a curve but a parameterized curve. More formally path p with endpoints A, B is a continuous map $[0, 1] \to S$ from unit interval to space S where A, B live, such that $p(0) = A$ and $p(1) = B$. While in the standard CM *time* features as a universal or "absolute" external parameter, in Homotopy theory the corresponding parameter is used internally for each particular path as just explained. Compare the notion of proper time along a worldline in General Relativity. The difference between these two ways of thinking about time is quite substantial but it plays no role in my analysis of *Venus*. The idea of identification of moving objects via continuous trajectories belongs to BKS and by itself does not involve the notion of external universal time. This latter notion is specific for CM, and its analysis is out of the scope of this paper. Historically BKS can be safely attributed to all pre-relativistic astronomy including ancient astronomical theories, which don't involve anything like the Newtonian concepts of absolute space and time.

[12]The proposed HoTT-based reconstruction of Frege's $Venus$ example may not capture Frege's volatile notion of sense in general. This notion may comprise more than HoTT in its existing form is able to detect. For example, arithmetical propositions

7. Are Intensions Real?

Recall Frege's question: What is the difference between the sense of proposition (1) $MS = MS$ and the sense of proposition (2) $MS = ES$? It appears to be in accord with Frege to assume that senses of propositions depend functionally on their corresponding proofs (even if proofs and senses are not exactly the same). Then our reconstruction of *Venus* allows for a precise mathematical answer to Frege's question: while the (unique) proof of (1) is trivial loop $refl_{MS}$, the proof of (2) is a non-trivial path p. In both cases a given proposition has a single proof. However these two proofs essentially differ not only in their intuitive "sense" but also in their geometric representation.

Let us now turn to some ontological issues. Albeit the concept of proof is epistemic *par excellence*, the HoTT-based reconstruction of *Venus* makes it clear that proofs in the standard proof-theoretic semantic of MLTT should not be necessary thought of as purely mental constructions. Thinking about such proofs as *truthmakers* opens a way to various forms of *truthmaker realism* [22]. Whether or not one takes *Venus* and/or its trajectory p to be real entities depends, of course, on a particular ontology that one may associate with CS or another theory supporting the relevant astronomical observations. In particular, CS allows for a 4-dimensional ontology where atomic entities are points of Classical aka *Neo-Newtonian* space-time (*Sklar:1974*, p. 202 ff). In this ontological framework p, seen as a world-line, qualifies as a full-fledged entity while the moving object *Venus* is its momentary slice. I shall not discuss here details of this and rival ontologies but rely on the fact that p of our example allows for natural realistic interpretations.

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(1) '2+2 = 4'

and

(2) '4 = 4'

arguably have *different* senses. However the standard Peano-style formalization of arithmetic used in HoTT treats both equalities (1) and (2) as definitional, see [16], p. 36 ff. At the same time, given Frege's specific view on arithmetic as a part of logic developed in his [**?**], it is not obvious to me that the view that (1) and (2) have one and the *same* sense is indeed untenable in a Fregean conceptual framework. In any event this controversial issue does not have a bearing on my following argument.

According to Frege, senses should not be thought of as psychological entities belonging to individual minds ([5], p. 38-39). However he suggests that senses wholly belong to human collective memories stored in existing natural languages. The only way in which a given sense can be possibly related to the non-human parts of our world, according to Frege's account as I understand it, is via the reference (if any) of the corresponding linguistic expression. For example English word "apple" has a sense, which belongs to this language (and arguably is shared by other natural languages) and a reference, which is a real thing that may exist independently of any linguistic and other human activities. English word "unicorn" equally has a sense but has no reference; so this particular sense is detached from any non-human reality.

The above is a rough interpretation of Frege's view but it points to a common idea about linguistic meaning, which is worth being considered here. Since Frege's concept of sense and the logical concept of *intenision* are closely related (see the end of Section 2 above), the standard examples of so-called *intensional contexts* apparently provide a further linguistic support to this idea. Such examples always have to do with intentions, beliefs, knowledge and other human-related issues. So these examples square well with Frege's view according to which propositions $MS = MS$ and $MS = ME$ have "different cognitive values" because their senses are different - in spite of the fact that their reference (truth-value) is the same.

Our analysis of *Venus* suggests a revision of this view. Since proofs are constituents of senses (of propositions), and since these proofs admit realistic interpretations, such realistic interpretations may extend to senses. What I have in mind is <u>not</u> a justification of some form of Meinongian existence of unicorns but rather the view that the distinction between the sense and the reference of a given linguistic expression must be freed from all ontological commitments altogether. The idea that the reference is the only linguistic anchor that links human languages and the human cognition to non-human realities is hardly justified. Sense and reference and their logical counterparts such as intensions and extensions of concepts all make part of (various versions of) our conceptual apparatus. How this apparatus connects us, humans, to non-human realities is a question, which cannot be answered only by means of logical and conceptual analysis.

I submit that behind the view on meaning, which I purport now to criticize, is the following strong ontological assumption:

(OE) *All real entities are individuals.*

For further references I shall call this assumption the *ontic extensionality* or OE for short. The reason why I call this assumption *extensionality* becomes clear from a homotopical reconstruction of Frege's distinction between sense and reference, which generalizes upon the above reconstruction of *Venus* as follows. *References* are point-like individuals belonging to classes of alike individuals, which constitute *extensions* of their corresponding concepts. *Senses* are higher-order homotopical structures, which involve spaces of paths and their homotopies (including higher-order homotopies), and constitute *intensions* of the same concepts. As we have already seen, in the *extensional* version of HoTT the higher-order part of the structure is truncated. Hence the name for OE, which allows the truncated higher-order part of the structure to have an epistemic and cognitive value but includes in the ontology only its basic 0-level part.

From this point view it appears reasonable to claim that talks of apples, of unicorns, of Bucephalus and of Alexander the Great have the same logical form, so the words "apple" and "unicorn" both have a sense and a reference. By the reference of "unicorn" I understand here a fictional individual. Propositions about apples and unicorns may well allow for the same forms of truth-evaluation. The difference between merely fictional, legendary and real entities concerns material (contentful) rather than formal features of truth-evaluation. There is no way to distinguish between a fiction, a legend, and a historical fact on purely formal grounds [13].

[13]The Bucephalus example demonstrates this particularly clearly. Bucephalus is a legendary horse belonging to Alexander the Great. According to the legend Bucephalus was born the same day as Alexander and, according to a particular version of the same legend, he also died the same day as Alexander. I don't know about a verdict of today's historical science as to how much of this story (if any) is a historical fact

I can see no a priori reason for assuming that a part of the homotopic structure is more apt to represent reality than any other. For that reason I don't take OE for granted. Moreover that our reconstruction of *Venus* suggests that terms of 1-types (paths) allow for a realistic interpretation as well as terms of 0-types (points). However in the next Section we shall see that the situation is not so simple, and that BKS is compatible with OE after all.

Concluding this Section I would like to remark that OE goes along the view according to which the Classical first-order logic (FOL) should be seen and used as the basic logical tool for scientific reasoning. In this context the suggestion to drop OE and allow for higher-order entities sounds a part of an argument in favor of a higher-order system of logic with a standard class-based semantics. MLTT and HoTT indeed qualify as higher-order systems in a relevant sense but the homotopical semantic used in HoTT is not standard. In HoTT higher types are formed not by the reiteration of the powerset construction (i.e. not by considering classes of classes of ... of individuals) but in the geometric way, which has been briefly explained in Section 5 above. Our homotopical reconstruction of *Venus* given in Section 6 demonstrates how the geometric semantic of HoTT helps one to use this theory as a tool for mathematical modeling *in* science, not only as a tool for a logical analysis *of* science. I believe that this dummy example points to interesting theoretical possibilities in mathematical physics. For serious attempts to use HoTT and its logical structure in physics see [20], [19].

8. Basic Kinematic Scheme

Here I supplement the homotopical reconstruction of *Venus* from Section 6 with a similar reconstruction of the Basic Kinematic Scheme (BKS), which captures the usual idea of moving particle. The kinematic space K, in which MS and ES live, allows for multiple paths (trajectories) sharing their ending points. I think about K not as a vehicle of moving particles but rather as a collection Pt of such particles provided with an additional

and how much of it is a fiction. I don't believe that any advance in formal logic may help for answering this question.

structure, which represents their motions. The motions are represented by paths between the particles as in the *Venus* example. The additional structure is that of groupoid of paths over Pt. I do not include into K homotopies of paths beyond the trivial ones because such things play no role in BKS. Paths in K are assumed to be reversible and composable by concatenation; the composition is associative [14]. So in terms of HoTT K can be described as a 1-type. Pt is its underlying 0-type obtained from K via (0-)truncation.

Let me now briefly repeat our homotopical reconstruction of *Venus* in this slightly extended context. We take two points MS, ES in Pt and hence in K, and consider path space $MS =_{Pt} ES$, which is a subspace of K. Then we find in $MS =_{Pt} ES$ a particular path p, which serves us as a proof of identity $MS = ES$. The extended context allows us now to notice a fundamental feature of BKS, which so far remained out of the scope of our analysis. It concerns the distinction between *possible* and *actual* paths. All terms of path space $MS =_{Pt} ES$ can be described as *possible* paths, while the observed path p is the unique one, which also qualifies as *actual*. The distinction between possible and actual paths is essential for BKS because it is unthinkable - or more precisely unthinkable within BKS - that the same particles follows two different paths simultaneously. Groupoid K by itself does not reflect this feature, so we need an additional effort for taking it into account [15].

[14]In the usual Homotopy theory the composition of paths in a given space S is defined only up to homotopy; in order to define such an operation one is obliged to provide an appropriate homotopy aka reparameterization by hand. Since in HoTT homotopy types are primitive objects this issue is treated a bit differently in this case. I shall only explain how it works in our reconstruction of BKS. We begin with an abstract groupoid K without assuming any ambient space S in advance, and then see how much of BKS we can recover in this way. This allows us to call the composition of paths in K concatenation without talking of homotopies

[15]In order to make this feature more evident in Frege's setting let us assume that MS and ES are observed during a single 24-hours period of time and that definitions of these objects are designed accordingly. Suppose one detects p, which is the trajectory of *Venus* during the time span between the morning and the evening of that day. On this basis one concludes that MS is nothing but *Venus* observed in the morning and ES is nothing but the same planet observed in the evening. It is unthinkable in such a situation that

Let us see how certain possible paths in K are transformed into actual paths. I assume that actual paths are closed under composition and thus form a subgroupoid $A \subset K$. The structure of A can be fully reconstructed on the basis of only two plausible principles:

(1) (the uniqueness of actual path)
A is *thin*, that is, has at most one path between any two given points.

This principle excludes non-trivial loops (since every point already has a trivial loop, which is its reflection) and all cycles (since all triangles commute). So A is a preorder.

(2) (equivalence)
A admits no branching.

Branching is excluded because it would represent "splitting identities" like in the popular *Ship of Theseus* example. Such anomalies can considered elsewhere but not in BKS, which takes it for granted that particles remain intact during their motions. More formally (2) says that the identity relation represented in A by a set of paths, is an equivalence. Together the two principles imply that A is a preorder of a trivial spaghetti-like sort. Each particular noodle of A can be called a *worldline* of its corresponding particle (point). Since such a noodle is contractible into a point, A and Pt are homotopically equivalent, i.e., represent the same 0-level homotopy type $A \simeq Pt$. The operation that takes space K of *possible* trajectories into space A of *actual*trajectories of the same points can be now formally described as 0-truncation.

It is remarkable that we obtain here the familiar picture of 4D spatiotemporal "worms" or "noodles" on purely formal grounds without referring to complicated issues concerning space, time and spacetime [16]. We have used for this only a tiny fragment of HoTT, which can be easily translated into a programming code, and two additional principles, both of

there could exist a different trajectory $p' \neq p$ followed by the same moving object during the same period of time.

[16]Since we are talking about the Classical Mechanics but not about the Relativistic Mechanics, the relevant notion of spacetime is that of the *Neo-Newtonian* spacetime, see [21], p. 202 ff. Such an anachronistic notion of spacetime shares a number of its basic constituents with its Relativistic successor. It goes without

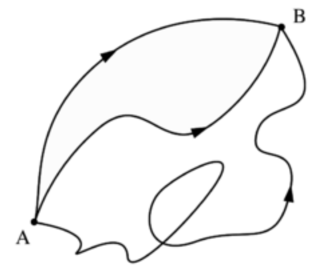

FIGURE 2. Quantum Paths

which have formal character and can be also straightforwardly expressed in HoTT. At the same time I would like to stress that the geometric intuitions associated with HoTT greatly helped us to apply this formal tool in our analysis.

The above analysis of the BKS appears to be an appropriate starting point for building a Quantum counterpart of this conceptual scheme. From the homotopical point of view taken here there is nothing impossible or unnatural in the idea that a given particle may follow multiple trajectories simultaneously as this is assumed in the Feynman path integral formulation of Quantum Mechanics:

One may rather wonder *why* BKS requires truncation of K into A. What is behind the traditional notion according to which the *actual* trajectory of a given particle during its lifetime is necessary unique?

In order to provide a tentative answer let me return to the issue discussed in the last Section. The above analysis of BKS apparently provides an additional evidence in favor of OE (notice extensionality). The intensional groupoid structure of K represents *possible* trajectories of particles. But since in the real world each particle has a unique worldline the groupoid K is reduced (truncated) to the extensional set $A \simeq Pt$. Conversely, one may argue that OE in the given setting implies the uniqueness of actual path. However

saying that the difference between the two spacetime concepts is essential. However in the present analysis I focus on what they share in common.

in order to make BKS and OE compatible one should understand the modal property of being *possible* (for paths) in purely epistemic terms - say, as a lack of knowledge about the actual trajectories. If, on the contrary, one thinks about possible paths in K as physically real then one violates OE anyway. This latter view is not wholly unreasonable and the Quantum Mechanics where the truncation $K \to A$ does not universally apply, provides additional reasons for taking it seriously. I stop here and leave an attempt to develop a HoTT-based theory of identity for Quantum Mechanics for a different occasion.

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