Venus Homotopically

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Idea

To use Homotopy Type theory (HoTT) as a novel formal framework for an analysis of Frege's classical *Venus* example.

Frege uses this example for developing his theory of identity and for justifying his distinction between the *sense* and the *reference* of a given linguistic expression.

Intended Aim

To elucidate logical and epistemological implications of HoTT with its non-trivial construal of identity by introducing this theory into a broader historical and philosophical context.

Frege's *Venus* still remains central in the continuing logico-philosophical discussion on identity and related issues.

Result (beyond the intended one)

A HoTT-based reconstruction of the Basic Kinematic Scheme applied throughout the Pre-Relativistic and the Pre-Quantum Physics.

This reconstruction appears to shed a new light on the distinction between Classical systems and Quantum systems.

Outline

Identity according to Frege

Die Identität ist eine so bestimmt gegebene Beziehung, dass nicht abzusehen ist, wie bei ihr verschiedene Arten vorkommen können. (Grundgesetze 1893)

Identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur.

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- Frege's Abstraction : $a = b$ where this time the two symbols denote a new abstract object called *direction*
- Geach's Relativization : $a =_{direction} b$ but $a \neq_{line} b$ (a polymorphic typing ?)

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- (2) *Venus* = *Morning Star* (convention)
- (3) *Morning Star* = *Evening Star* (non-trivial fact)

Observation

- (1)-(3) have different "cognitive values"
- Substitution *salva overstate* does not apply universally in epistemic and other *intensional* contexts. Ex. :
  Frege *knows that Morning Star* is *Evening Star*

Problem

Where the difference between cases (1)-(3) comes from and how Logic should account for it ?

Mathematical example

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- 2 = the only even prime (nearly trivial : follows from the definitions of even and prime)
- 2 = the maximal natural n such that $x^n + y^n = z^n$ has a natural solution (highly non-trivial)

Frege's solution for *Venus*

Double Semantics : Sense and Reference

Reference

- $Reference(V) = Reference(V)$
- $Reference(V) = Reference(MS)$
- $Reference(MS) = Reference(ES)$

Sense

- $Sense(V) = Sense(V)$
- $Sense(V) = Sense(MS)$
- $Sense(MS) \neq Sense(ES)$

Double Semantics for propositions

- $Ref(V = V) = Ref(V = MS) = Ref(MS = ES) = truth$
- $Sense(V = V) = (?) Sense(V = MS) \neq Sense(MS = ES) = ?$

Intension and Extension

- *Sinn*/*Sense* ∼ *Inbegriff*/*Intension*
- *Bedeutung*/*Reference* ∼ *Umfang*/*Extension*

Ontic Existensionality (OE)

- Sense / Intension is epistemic
- Reference / Extension is ontic

Weak OE

All entities are individuals

but not all individuals are entities

Definitional aka judgmental equality/identity

$x, y : A$ (in words : x, y are of type A)

$x \equiv_A y$ (in words : x is y by definition)

Propositional equality/identity

$p : x =_A y$ (in words : x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER does not follow from other principles of MLTT ;

Uniqueness of Identity Proofs (UIP) conjecture has been refuted in 1999 by Hoffman and Streicher with their groupoid-based model of MLTT

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x=_A y} y'$
- $\ldots$

HoTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y ; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x=_A y} y'$
  x'', y'' are homotopies between paths x', y' ; $x' =_{x=_A y} y'$ is the space of all such homotopies
- ...

Cummulative Hierarchy of Homotopy Types

- 0-type : points in space with no (non-trivial) paths
- 1-type : points and paths in space with no (non-trivial) homotopies
- 2-type : points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Truncation

n-type can be transformed into its underlying m-type with $m < n$ by forgetting (trivializing) its higher-order structure of all levels higher than m.

Quine 1969 on "no entity without identity"

[W]e may have in the bulk term a relic, half vestigial and half adapted, of a pre-individuative phase in the evolution of our conceptual scheme. And some day, correspondingly, something of our present individuative talk may in turn end up, half vestigial and half adapted, within a new and as yet unimagined pattern beyond individuation. Transition to some such radically new pattern could occur either through a conscious philosophical enterprise or by slow and unreasoned development along lines of least resistance. A combination of both factors is likeliest [...]

Does HoTT provide such a pattern "beyond individuation". At the very least it provides a novel pattern of individuation where identities are allowed to *evolve*.

Here we start with certain *fixed* (definitional) identities but do not exclude the possibility that in the process of reasoning certain fixed identities may collide.

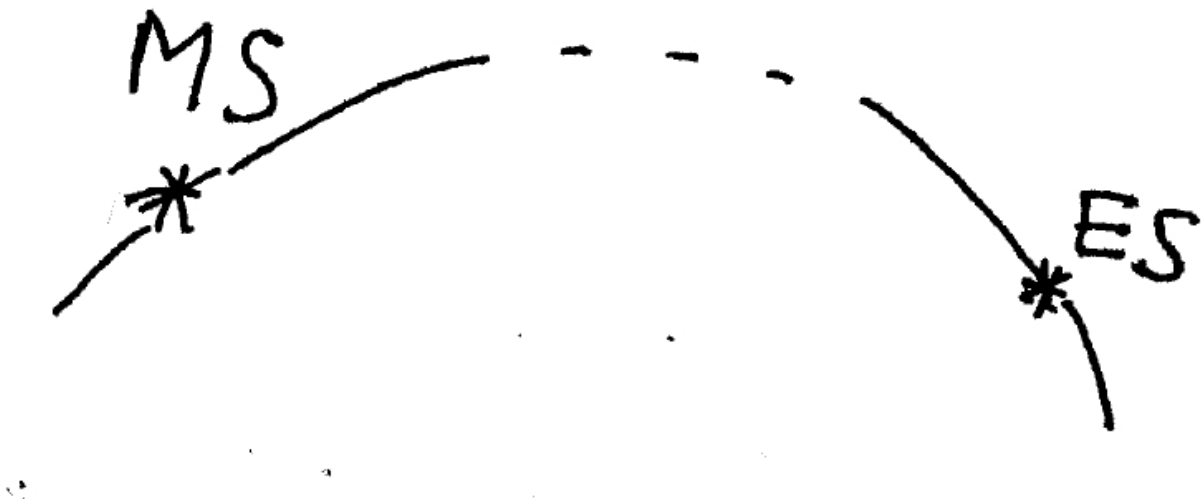
This appears a rather innocent modification of the standard extensional individative scheme. One may further speculate about a scheme which applies no fixed identity at all. It would not use definitions; every identity would be propositional and non-trivial. Still, even the "tamed" evolving identity such as in HoTT is very rich mathematically.

The *Venus* example demonstrates that *evolving identities* can look more familiar than Quine imagined when he talked of "half-entites inaccessible to identity" ....

Venus does not belong to one's language !

The discovery that the rising Sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even today the identification of a small planet [i.e., an asteroid - *A.R.*] or a comet is not always a matter of course.

MS=ES : a geometrical (form of) proof



Path p witnesses that the *Morning Star* and the *Evening Star* are the same planet *Venus*.

MS=ES : a syntactic (form of) proof

- $MS, ES : Pt$
- $p : MS =_{Pt} ES$, where $MS =_{Pt} ES$ is a path space

Notice that both *MS* and *ES Venus* have non-trivial internal structure. These objects are construed on the basis of separate observations and theoretical reasoning made in different places at different times by different people. Cf. Frege's remark about the Rising Sun. However in this given reasoning the internal structure of *MS* and *ES* is not taken into the account : these objects are introduced by definitions.

Modality Issue

$MS =_{Pt} ES$ is a space of all possible paths. But p is the unique actual trajectory of *Venus*. It is unthinkable (as far as the Pre-Quantum Mechanics is concerned) that *Venus* follows two different paths simultaneously. This fundamental principle of BKS does not follow from HoTT and should be considered as a further principle.

Groupoid of possible paths

Let K be groupoid of possible paths of particles Pt. Paths identify the points as in the *Venus* case. Let all homotopies be trivial (no second- and higher-order structure).

I take here a relationalist approach and do not assume that particles Pt and their paths live in some ambient space S. I rather purport here to construct such a space by talking about particles and their paths.

Groupoid of actual paths

Let $A \subset K$ be groupoid of *actual* paths with the following properties :

- (1) A is *thin*, that is, has at most one path between any two given points (the uniqueness of actual path)
- (2) A admits no branching (identity is an equivalence)

Worldlines

Hence A is pre-order of trivial spaghetti-like sort, where every noodle is contractible into a point : $A \simeq Pt$

Notice that the familiar worldline picture has be just obtained on formal grounds (HoTT + principles (1)-(2)) without references to usual issues of space, time and spacetime.

Modal Truncation in BSK

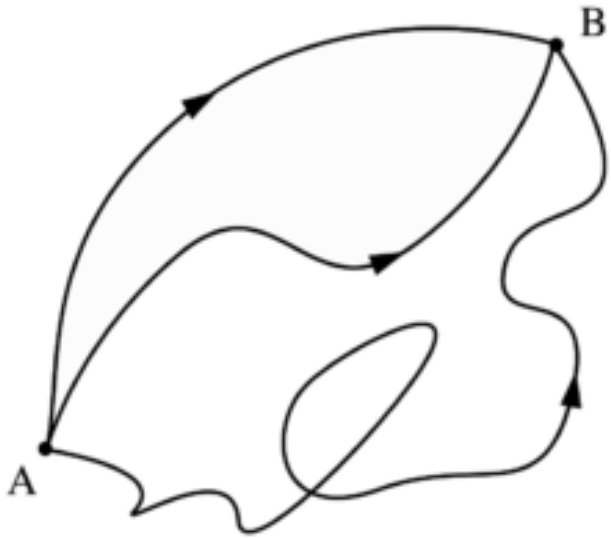
A is obtained from K through 0-trancation, i.e., by forgetting the higher-order path structure.

Why this truncation ? Assuming that only actual paths qualify as existent one may claim that BSK involves the weak OE (there exist only particles but not their paths).

Are intensions real ?

But one may also argue that possible paths represent real physical possibilities and are equally existent.

Quantum Paths



In the Quantum case the BSK-style modal truncation, generally, does not apply.

Open Problem

From the HoTT perspective the Quantum case looks more simple and more "natural" than BSK. To develop a HoTT-based theory of Identity for Quantum particles remains an open problem.

THE END