Category theory and Mathematical Structuralism

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Abstract:
Category theory doesn't support Mathematical Structuralism but suggests a new philosophical view on mathematics, which differs both from Structuralism and from traditional Substantialism about mathematical objects. While Structuralism implies thinking of mathematical objects up to isomorphism the new categorical view implies thinking up to general morphism.

It has been argued in recent discussions (see Awodey 1996 and following discussion in *Philosophia Mathematica*) that Category theory provides a strong support for Mathematical Structuralism. Proponents of this view consider a category as a structure and then stress the fact that most of other structure types like groups, topological spaces, vector spaces, etc. "form" categories. So one gets categories of groups, topological spaces, etc. The structure type of categories needs not to be excluded from this list since categories form a category (the category of categories) in the same way in which do groups and topological spaces. This allows for considering categories as "very general" structures and arguing that category-theoretic mathematics is essentially "structural". So the fact that "categories are everywhere" provides, so the argument goes, a strong support for the thesis according to which all of mathematics is essentially structural, i.e. for Mathematical Structuralism.

I shall argue here that the above argument is wrong, and that Category theory doesn't support Structuralism but suggests a different philosophical view which, however, doesn't reduce to what is usually taken to be the opposite of Structuralism, namely some version of Substantialism about mathematical objects.

First of all, let me show that the view on categories as structures doesn't fit most "workable" categories like categories of groups, topological spaces, vector spaces, etc... . For unlike "individual" groups, etc., categories of such things are not *sets* provided with some structures. "All" groups, topological spaces etc. form proper classes rather then sets. For this reason the categories in question are called "large". Small categories alone cannot demonstrate the full power of Category theory and its impact to other domains of mathematics. But large categories cannot be construed by the usual structuralist method of providing a given set with an appropriate structure.

One may argue that the "size difficulty" just stressed is not particularly worrying, and so instead of sets one can use proper classes without changing the method. Perhaps the real difficulty here is indeed not about size. But the suggested replacement of sets by proper classes is not compatible with the usual structuralist method in any event. Here is why.

All existing concurrent notions of mathematical structure (Heller , forthcoming) agree on this general point: a structure is a thing determined up to isomorphism. But the usual notion of isomorphism is not quite appropriate for categories (Gelfand&Manin 1996). Without going to technical details this can be explained as follows. Given a group G one may consider an isomorphic group G' as just "another copy" of (group of type) G. The ambiguity between what is a token and what is a type here is systematic but harmless. But if **G** is the category of (all) groups there is no sense in which it might have another "isomorphic copy" **G'**. For **G'** cannot be anything but another category of all groups. But we agreed that all groups are already in **G** and so "nothing remains" for **G'**. However naive this argument might be it shows that cases of G and **G** are quite different.

My proposed solution of this puzzle is this: categories are not structures but things which are "more general" than structures. The idea behind the notion of mathematical structure is to construe mathematical objects falling under a given concept (think about groups) up to isomorphism. The idea behind the notion of category is to construe objects falling under a given concept "*up to general morphism*" rather than up to isomorphism. The expression "up to general morphism" is not particularly fortunate and I use it here only for stressing the difference between the categorical and the structural approaches. There is, of course, no sense in which different groups are all the same. Nevertheless all of them belong to the same category.

The structural and the categorical approaches are not incompatible. Construing groups, topological spaces, etc. as structures one can consider then categories of these things. In this case the structural approach is given a priority: one gets structures first and make them into categories later. Historically people started to use categories in this way. But this is not the only possible way to proceed. Remark that an "individual" group can be seen as a particular category (with a single object) rather than as a structure. Groups construed as categories can be equally made into one category of groups. Similar constructions are possible for topological and vector spaces. Some of these constructions turn to be highly non-trivial. Think of Grothendieck topology for example.

Thus categorical thinking in mathematics doesn't reduce to structural thinking. The notion of category generalises upon that of structure just in the same sense in which the notion of general morphism generalises upon that of isomorphism. It is still common to say in mathematics that morphisms "preserve a structure" in the general case. But this is a misleading expression, which suggests thinking about all morphisms as if they were isomorphisms. Think about a group homomorphism sending a given group into the trivial group consisting of single element with identity. There is a sense in which this homomorphism "forgets" or “kills” rather then “preserves” the structure of the given group. Nevertheless this forgetful mapping is a group homomorphism, it transforms the given group into another group.

Thus the categorical "thinking up to general morphism" is more general and in any rate quite different from the structural "thinking up to isomorphism." All consequences of this generalisation are still to be explored. In my talk I shall elaborate on some of them.

Bibliography:

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