A Philosophical Reflection Concerning the Applicability of Mathematics in Physics

On the "unreasonable effectiveness" of mathematics in the natural science

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Eugene Wigner closes his famous paper "The Unreasonable Effectiveness of Mathematics in the Natural Sciences"[1] with the following assertion: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

I believe that Wigner' bafflement vis-à-vis the usefulness and "miraculous" adequacy of mathematics, its language and concepts, for the formulation of the concepts and laws of physics, and, as Mark Steiner later emphasized, sometimes for the discovery of these laws as well, betrays hidden presuppositions which, by remaining hidden, block the way to the adequate understanding of what should be straightforward scientific methodology.

My goal here is to bring to light these presuppositions and offer alternative views, particularly on the nature of mathematics and physics, which I believe go a long way into dispelling the fog of mystery that surrounds the efficacy of mathematics as an instrument of scientific inquiry. Either mathematics and physics are what Wigner thinks they are, and the applicability of mathematics in science constitutes indeed a "miracle", or miracles do not happen, and Wigner's views on the nature of mathematics and physics are incorrect.

At the heart of Wigner's bafflement lies the presupposition that whereas mathematics is essentially concerned with its own concepts, devised for the most part without any particular regard for empirical reality, physics is concerned with a world, the physical world, conceived as an independent realm of being existing in itself and completely determined once and for all. Wigner carves an abyss between mathematics and physics and it would be surprising if he did not fall into it. There is indeed absolutely no reason why man-created mathematics should have any relevance for the description of man-independent empirical reality, to the point of being *essential* for even the *formulation* of its laws.

It is unquestionable that mathematics is a creation of man. Although some mathematics is invented for the purpose of coping with some aspects of human experience (Wigner mentions geometry, although practical geometry, or practical arithmetic, are not sciences, but

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[1] *Communications on Pure and Applied Mathematics*, 13: 1-14, 1960.

technologies, situated at a pre-theoretical level of involvement of men with geometrical and arithmetical forms), or the conveniences of science, mathematical concepts in general are free creations which answer, for the most part, to mathematical conveniences only.

Physics, on the other hand, is concerned with an external world, and if the applicability of mathematics in physics is to cease to be a mystery physical reality cannot be what Wigner thinks it is, an *independent* reality existing *in itself* but "miraculously" willing to open its secrets to mathematical concepts, to the point of *demanding* mathematics for its laws to be expressed. If mathematics is part of the very fabric of nature, but happens to agree with man-made mathematics, which is seldom created by *observing* nature, then there must be a link of some sort, causal or otherwise (for example, pre-established harmony), between human mathematics, a cultural product, and the mathematics of nature.

Some people have advanced a naturalistic explanation for the "miraculous" appropriateness of human mathematics for the description of nature thus: since man was naturally selected to survive in this world it is not at all surprising that man was also naturally selected to create instruments for understanding how this world functions. The problem with this line of reasoning is that among the strategies selected for the survival of men until the age of reproduction does not include the understanding of, say, the laws of quantum mechanics. On the contrary, the concepts necessary for the adequate explanation of quantum phenomena conflicts with those developed for surviving in the world (for instance, the concept of the trajectory of a body and the determinacy and unicity of trajectories given the dynamical situation).

My approach here is a different one. Unlike Wigner and others, who never question their realist, empiricist presuppositions, I take a transcendental idealist perspective of the type proposed by the phenomenologist Edmund Husserl, particularly in his *The Crisis of European Sciences and Transcendental Phenomenology* (1954/1936), where the intentional genesis of the concept of empirical nature proper to modern science as created by Galileo and others from the XVI century on is scrutinized. I claim, in agreement with Husserl, that the concept of nature prevalent in the modern mathematical science of nature is an *intentional construct* elaborated for specific methodological purposes. The scientific concept of empirical nature of modern science is an abstract and idealized representation of *some* aspects of our immediate perceptual experience of nature *specifically designed for mathematical intervention*.

Unlike Wigner, I do not presuppose that science provides a direct, non-intermediate description of nature – a view whose corollary, and with it the "mystery" of the unreasonable effectiveness of mathematics in science, is that the mathematics that science finds *essential* for the description of nature must ipso facto be an *essential* aspect of nature, something intrinsic

to it. I see three layers of reality where Wigner apparently sees only one; there is, I believe. Nature "out there", somehow causally responsible for a second layer of reality, perceptual nature, which is either directly grasped through the perceptual systems or somehow inferred from perceptual data with the help of our theories of nature (the part of perceptual nature which is not directly perceived is nonetheless supposed to be, in principle at least, accessible to immediate experience), and, finally, the mathematical construct by which science represents *some* aspects of perceptual nature. I will call these levels, respectively, *Nature*, *perceptual nature* and *physical nature*, by which I understand the conception of nature of modern physics (from the XVI century on). Perception cannot reach physical nature; reason only – and with it, mathematics – has access to it. But we must refrain from committing the Platonist mistake of reversing ontological priorities, by enthroning physical nature as the *true* nature, identifying it with Nature herself, and perceptual nature as only an imperfect glimpse of a reality that is forever out of adequate grasp. The primary object of physics is physical nature, and through it perceptual nature, which is how Nature presents herself to us. Mathematics is the language of physics, but the distance – the intentional distance, I may say – between physical nature and Nature herself blocks the way to the migration of mathematics from theory to reality.

I emphasized before that physical nature is an abstract and idealized version of *some* aspects of perceptual nature, and I will be more explicit about this below, but it is not difficult to guess what these aspects are. Perceptual nature presents herself as a combination of content and form; although, contrary to logical empiricist tenets, I (or Husserl) do not believe that elementary sense impressions of the sort "this red patch here now" are the sole immediate given of perceptual experience, I assume that the hyletic, material content of experience can be ultimately reduced to them. Perceptual experience is always constituted by matter, ultimately reduced to sense data, and form – and it is not clear to what extent the forms discernable in perceptual experience are intrinsic to Nature or, contrarily, imposed on sense data by intentional operations of perceptual systems – but only the forms perceived in experience can expect to find a place in physical nature. In this sense, physical nature is a formal-abstract reduction of perceptual nature. In its formal aspects, perceptual nature is already, to some extent, mathematical or proto-mathematical, but by *idealization*, i.e. by an *intentional process of exactification*, the forms perceived in experience become fully mathematical; these idealized forms constitute the original community of denizens of physical nature, the material content of which being either dismissed as essentially subjective or preserved by being given objective mathematical, that is, *formal* representatives.

But, and this is very important, physical nature can be arbitrarily enriched by forms that are *not* the idealized versions of forms immediately given in perception, introduced therein in

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order to more adequately handle mathematically those which are. These forms can correspond to forms in principle perceivable in experience, being then idealized versions of those, or be only purely mathematical accretions to physical nature required on grounds of theoretical reason. Mathematically enriched enlargements of what we could call the *primordial physical nature*, that is, the physical nature that corresponds to *actual perceptual experiences*, provide in general more adequate contexts for higher-level mathematization. Mathematical manipulations in the theory, referring either to primordial physical nature or its formal enlargements, may point out to facts that are in principle perceivable, *but only indirectly and exclusively with respect to their form*. *Thus mathematics play a heuristic role in science*. The particular material content of mathematically predicted phenomena, however, may not be a priori determinable if involving mathematical terms which do not have predetermined interpretations, or univocally determinable even in cases where there is a semantic context available for interpreting all the mathematical terms involved. Finding a context of interpretation in perceptual nature or a particular interpretation in a pre-given semantic context which conveniently idealized makes a determinate mathematical prediction true counts as an empirical verification of this prediction. But, most importantly, this is *not* a task for mathematics and depends heavily on the scientist's ingenuity.[2]

The possibility of arbitrarily extending physical reality into more and more mathematically sophisticated domains accounts for the extraordinary power of mathematics in our theories of nature, both descriptively and heuristically. But we must keep constantly in mind that it is not Nature herself, standing out there, or even perceptual nature that mathematized physics is describing, but a surrogate of the latter that only indirectly, and only by retracing backwards the steps of the intentional constitution of physical nature, refers to the former.

This introduction presents in a rough sketch the main points of the alternative view that I want to offer in contraposition to Wigner's "naïve" approach which, I believe, is the sole reason why the qualification "unreasonable" is connected with the effectiveness of mathematics in the empirical science. I offer a more detailed analyses of these views below, clarifying, in opposition to Wigner's views, what I take mathematics and physics to be, and more importantly, why the former can be so *essential* for the latter.

Mathematics: For Wigner, mathematics is a science of concepts, which are either devised for "describing" entities of experience (as, for instance, in geometry) or freely invented, in which case they obey only the rules that were freely created for operating with them. The

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[2] Dirac's mathematical manipulations, for example, that supposedly predicted the existence of the positron could have been interpreted differently, an electron moving backwards in time, for instance.

latter are more common "in advanced areas of mathematics, which play such an important role in physics".[3] This characterization can, I believe, be accepted without much reservation. The role concepts play in mathematics, however, deserves closer scrutiny.

Concepts are tools of the understanding for categorizing *objects* (intuitions, perceptual data and similar *individuals*) in order for them to be *thought*. According to Kant, we think through concepts (*KrV* A19/B33). So, concepts must be able to circumscribe well-determined realms of objects (or whatever counts as *given* to understanding). Sometimes objects are given independently of concepts, which are abstracted from them and characterized through *common notes*. But sometimes objects are *given* through concepts, which are both *means of donation* of and *tools for thinking* about objects. In mathematics both cases occur. Theorizing about a determinate domain of objects requires a language in which objects and concepts are denoted by, respectively, nominal and conceptual terms. Explicit characterization of concepts also require convenient linguistic and conceptual apparatuses; a conceptual characterization is *adequate* if it succeeds in singling out descriptively the domain (or extension) of the concept, either the objects from which the concept is abstracted or those the concept purports to characterize. The explicit characterization of a concept should, then, at least ideally, univocally determine the domain of the concept.

The categories of mathematics are essentially those of object and relation; mathematical descriptions are essentially descriptions of objects in relation, linguistically expressed, respectively, by nominal and relational terms. Mathematical descriptions can only hope *to fix* a domain of reference by establishing relations among objects of the domain and relations defined therein (in particular by means of properties, a particular type of relations). A typical characterization of a mathematical concept is that of finite cardinal number by the second-order Dedekind-Peano system of axioms.

The problems is that *no* linguistic characterization of any concept whatsoever, particularly in mathematics, is ever adequate. Domains of objects can only be uniquely determined *intuitively*, that is, by the immediate presentation of their objects in person, so to speak; for example, in perception or some form of intuition. Of course, the intuitive realm of numbers is a model of the Dedekind-Peano system, but it is not the only one. Even though second-order arithmetic is categorical, we can always artificially concoct domains of number-

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[3] "[…] mathematics is the science of skillful operations with concepts and rules invented for this purpose. The principal emphasis is in the invention of concepts. […] whereas it is unquestionably true that the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular of the concepts which play such an important role in physics".

like entities isomorphic to the realm of numbers proper that also satisfy the axioms. In order to circumscribe the axiomatic characterization of the concept of number to its *intended* interpretation we must resort to the original donation of this domain, numerical *intuition*. By *purely descriptive* means, i.e. in *language* only, we cannot ever adequately characterize concepts.

This is relevant when considering concepts that are *not* abstracted from intuitively given objects but freely invented. As we saw, in this case, the concept itself is the means of donation of the objects that fall under it. Let's consider Dedekind-Peano axiomatic system from this perspective. Numbers are no longer simply *described* by the axioms, they are instead *defined* by them. The numerical domain is now, *by definition*, *the* domain of entities that satisfy the axioms. But as we've just seen, the axiomatic characterization is unable to single out a domain of well-determined objects, *the numbers*, and not mere number-like entities. The conclusion is that freely invented concepts cannot ever be donator of objects, if by objects we mean well-determined *individuals*. They can, at best, single out object-forms or, to use Husserlian terminology, *formal objects*, determined as to form but indeterminate as to material content.

But *there is* something that the Dedekind-Peano system succeeds in characterizing, namely, the way in which the objects in *any* of its interpretations relate to each other. I will call the properties of a structured domain of objects, that is, a system **A** = <A, R_i> formed by a collection A of objects and relations R_i defined therein, which only involve *these* objects *with respect to these* relations *structural* properties of the domain. Structural properties are expressible in any language that can be interpreted in the system **A** (and *only* in them); I will call these descriptions *structural descriptions*. They can be classified with respect to the language in which they are given, first-order, second-order, etc. Structural properties are *formal* in the sense that different structured systems can have the *same* structural property. The axioms of Dedekind-Peano express formal-structural properties that are shared by an entire family of isomorphic structured systems, and these only. I will call such a system a *structure*. A structure is, then, a *class of isomorphism* or, on a more philosophical perspective, an abstract entity, a *form*, which is indifferently embodied in any member of the class (and which can be intuited provided the necessary intuitive act is performed, *abstraction* in this case) or "emptily" meant by a *categorical* system of axioms. The Dedekind-Peano axiomatic system, in particular, characterizes the "numerical" ω-structure determined by the "successor" relation (which can be embodied in objectual domains that have nothing to do with numbers or the successor relation proper, hence the scare-quotes).

Conceptual descriptions in mathematics (but not only in mathematics) boil down to structural descriptions of the *extensions* of the concepts described. We often use concepts

assisted by intuition, which fix intended domains of application[4]; the fact, however, remains that descriptions of concepts, no matter how accurately carried out, are only formal-structural descriptions; they hold in the domains of the concepts envisaged but also in all domains isomorphic to them. This alone accounts for the range of applicability of mathematical concepts, which often extrapolate the context in which they were created.

In short, to the extent that mathematics involves language and linguistic characterization of its concepts, it must be content with expressing only structural properties of conceptually circumscribed domains of objects. It is then more appropriate to characterize mathematics as the science of structures or families of structures which are either abstracted from intuitively given structured domains or freely invented.

Physics: According to Wigner, "the physicist is interested in discovering the laws of inanimate nature". This pretty much characterizes physics for him; inanimate nature is submitted to *laws*, the role of physics is *to discover* these laws. This simple, apparently undisputable claim harbors presuppositions that are so ingrained in our way of thinking that are not even noticed. It is my belief that only by uncovering the many layers of hidden presuppositions beneath our conception of nature that we can hope to dispel the "mystery" that surrounds the effectiveness of mathematics in physics. Here are some:

1. The *realist* presupposition that nature is a *given*, which perception and understanding simply stumble upon.

2. The presupposition that nature is submitted to *laws*.

It follows from (1) and (2) that the laws of nature are *intrinsic* to nature and have *nothing* to do with *our* particular way of *experiencing* nature and *categorizing* our experience of nature. Nature is uncritically taken as a closed domain of being, completely determined in itself and submitted to laws, the task of the natural scientist being that of *discovering* the laws of nature.

Let's consider the first presupposition. Nature is a given which we apprehend in perception; we perceive *Nature herself*. According to this presupposition, perception has no part in shaping nature *as we perceive it*. Evidences, however, indicate that the perceptual systems play an active role in perception. Indeed, the first task of the senses is *to filter*; our eyes are sensitive to only a small fraction of the electromagnetic spectrum; our ears can perceive only a reduced range of frequencies, and so on. But, more importantly, the perceptual systems operate in association with pre-categorical, built-in intentional systems whose task is *to make sense* of sensorial impressions. Instead of an incoherent mass of data coming from the senses, we perceive objects, objects *in relation*, patterns of objects, phenomena, processes, etc. Perception

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[4] Bare intuition and conceptualization account, respectively, for reference and meaning.

is an intentional act, and nature *as perceived* – that is, *perceptual* nature – seems to bear to a considerable extent *our imprint*. If this is true, the immediate object of physics is not nature "out there", simply given to us – what I called earlier Nature herself –, but perceptual nature, intentionally elaborated by us, or still something erected on the basis of perceptual nature.

In fact, as history shows, the true object of physics, at least from the XVI century on, is not primarily *perceptual* nature, but a *mathematical* substitute of it which I call *physical* nature.[5] Regardless of whether perceptual nature already bears the imprint of particularly human forms of perceiving or else, whether perception simply mirrors nature, the fact is that we perceive patterns, forms, or structures in nature. Whether some of these structures, at least, are contributions of our perceptual system in the processing of the hyletic material coming from the senses or whether the structures we perceive are all intrinsic to nature, the fact remains that we perceive structures in nature. And who says structure says mathematics. Mathematics is the science of structures, either freely conceived or intuitively presented to consciousness; *the creation of mathematics as an intellectual endeavor so early in human history testifies to the ubiquity of structures in human experience, our capacity to consider them abstractly and our willingness to invent more*.

Some perceptual structures stand out so conspicuously that they invite mathematical investigation; for example, the proto-geometrical and topological structures of perceptual space. But, more often, the patterns that we perceive in nature can only become fully mathematical by being *idealized*; perceptual space, for example, only admit a geometric structure proper, including a metric structure, after being idealized into a manifold of mathematical points.[6] Privileging structures of experience that admit, either directly or upon idealization, a mathematical treatment was a novelty introduced in science at the beginning of modern times (by Galileo, Descartes, and Newton, among others); this *radically changed our conception of nature*, *physical nature, a mathematical manifold accessible only to mathematical reason* replacing *perceptual nature* as the immediate object of concern. Two intentional acts stand between perceptual nature and the conception of nature of modern physics, physical nature, *abstraction*, which brings relations and correlations, that is, form instead of content, to the focus of scientific interest, and *idealization*, which makes forms discernable in perception

[5] I here follow the lead of Husserl's seminal work *The Crisis of European Sciences and Transcendental Phenomenology* (Evanston, IL: Northwestern University Press, 1970; Engl. trans. from *Die Krisis der europäischen Wissenschaften un die tranzendentale Phänomenologie*, Husserliana vol. VI, The Hague: M. Nijhoff, 1954 (1936))

[6] See, for instance, Hermann Weyl discussion of the abyss between the intuitive (perceptual) continuum and the mathematical continuum in his *Das Kontinuum* of 1918 (Engl. Trans. *The Continuum*, S. Pollard and T. Bole (trans.), New York: Dover, 1994)

into mathematical forms properly (thus reducing forms which are actually *perceived* to rough *approximations* of supposedly *more real* mathematical forms).

The *matter* of perception (stuff like sounds, colors, etc.), however, is not simply dismissed but given objective representatives in an overall scheme of causations and correlations reducible to mathematical determinations. Colors, for instance, among other "secondary qualities", are taken as mere subjective impressions "caused by" a determinate type of radiation that shows up in the explanatory scheme of science as a mathematical "field" filling empty space and characterized by a set of mathematical parameters.

Let's now briefly consider the presupposition that nature is submitted to laws. Obviously, no amount of evidence can count as either a *proof* or a *disproof* of this presupposition; so, it cannot have the status of a scientific *hypothesis*. Particular laws can be hypothetical but the presupposition that nature abides to the rule of law cannot. But if not a hypothesis, what then? The answer is straightforward: *the lawfulness of nature is also a component of the intentional constitution of our scientific conception of nature, it is how we conceive nature to be*. The presupposition stands no matter how often we are disappointed in submitting nature to laws simply because science *depends* on it. In physical nature the laws of nature are, of course, mathematically expressed, and the concepts for adequately expressing them often require mathematics in an essential manner (the examples are legion, instant velocity, entropy, fields of different types, scalar, vector, tensor, observable in quantum mechanics, etc., etc.).

By ignoring the intentional process involved in the modern conception of empirical nature and by reinterpreting the historical development of this process as the unveiling of the *true nature* of nature (Galileo appearing in the process not as the inventor of a *method* for investigating perceptual nature, but as the discoverer of a *truer nature* behind the given of perception only accessible through mathematics, the famous Galilean metaphor about the book of nature) Wigner takes by a *given* what is only a *product*. The "mystery" of the applicability of mathematics in physics derives from this identification. Obviously, mathematics must have a major role in the investigation of the mathematical manifold that physical nature is conceived as being, but insofar as one sees physical nature as a given and mathematics as a product of human culture developed for the most part without paying particular attention to how nature is, one is inevitably faced with a puzzle. In fact, the applicability of mathematics in physics is only an aspect of a more general logical problem, interesting in itself, that of the applicability of

mathematics in mathematics, which boils down to the logical problem of how different structures and their theories relate to each other.[7]

The applicability of mathematics in physics: Wigner recognizes the following applications of mathematics in physics:

1) As a tool for drawing consequences within the mathematical theories of nature. This, of course, requires that nature has already been mathematized so theories of nature are mathematical theories. This, as Wigner says, is a straightforward use of mathematics in science that poses no problem.

2) As an essential conceptual instrument for the formulation of the laws of nature. For Wigner, this is how mathematics becomes master of the field. Wigner gives as example Dirac's formulation of the laws of quantum mechanics: there are two basic concepts, states and observables, the former are vectors in a Hilbert space, the latter are self-adjoint operators, the possible values of observations are proper values of these operators, and so on. The spaces of quantum mechanics are complex spaces and, as Wigner observes, the theory of complex numbers and analytic functions seem "destined to play a decisive role in the formulation of quantum theory". Wigner sees a "miracle" here, similar, he says, to the double "miracle" that nature is submitted to laws and that that the human mind has the "capacity to divine them".

Before proceeding, let's consider the miraculous character of the existence of laws of nature and our capacity to divine them. As we have seen above, the lawfulness of nature is an intentional aspect of the constitution of the concept of nature, not something that we discovered by observing nature (regularities that are effectively observed depend on a way of looking and a selection of what to look at and cannot per se *justify* the presupposition that nature's behavior accords to laws). Moreover, our modern conception of physical nature was devised so as to be accommodated within an essentially mathematical model of explanation, which fits our resources of explanation. Wigner sees "miracles" where he should see intentional action. The best explanation Wigner can offer of this "miracle" is to remember Einstein rather suspicious claim that mathematics is driven by a notion of beauty and the theories of nature that we are willing to accept are the beautiful ones. Of course, anyone who has so far attempted a characterization of mathematical beauty or elegance has failed, and, moreover, there are plenty of ugly, but nonetheless useful mathematics.

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[7] Husserl reserved a whole chapter of formal logic for this task, *formal ontology*, of which formal mathematics is a part (see in particular his *Formal and Transcendental Logic*)

But Wigner has a stronger claim, namely, that the language of mathematics is the *correct* language of nature. He gives two cases in which mathematics seems to offer the *natural* context for a law of nature to be expressed:

a) Newton's law of gravitation. After Galileo and others had established the law of falling bodies, Newton extended it to a law of gravitation which is *much more accurate* than Newton could have possibly expected given the experimental data of the time. Moreover, Newton's law depends essentially on non-intuitive higher mathematical concepts such as that of a second derivative. Of course, by choosing to express the law of falling bodies as a *mathematical* relation involving space and time (a quadratic function), Galileo chose a *style* of explanation (in particular, by mathematizing the continua of space and time *beyond the limits of accuracy of the means for measuring time and distance of the time*), and prepared the field for analogous explanations. Now, a body in orbit is a limit case of a falling body, and within the context of explanation inaugurated by Galileo it is to be expected that the trajectories of orbiting bodies (ellipses or conic curves in general) should belong to the same class of curves of falling bodies (parabolas). Newton genius revealed itself in offering a *unifying dynamical principle* to movements both in earth as in heaven and in the development of its theory well beyond the point where his predecessors (Boyle, Hooke, and others) had arrived.

Wigner calls our attention to the *accuracy* of the law of gravitation. By accuracy he means that the law holds beyond the limits of its evidential basis. The law is confirmed by much more refined evidence than that available at the time it was "discovered". By being an *ideal* law, involving, that is, idealizations concerning in particular the structure of space and time, Newton's law is, of course, like all laws of mathematical physics, accurate only up to certain limits of approximation. As our means for measuring physical magnitudes improve, "errors" vis-à-vis the law are more likely to appear. A law is more or less accurate depending on how soon "errors" are detected. *Any* empirical law is accurate until it isn't, and Newton's law is no exception. Being *ideal* laws empirical laws never perfectly *match* the *perceptual* data available and there usually is a range within which data can vary and still accord to the law. It is like drawing a line on the plane that crosses a number of pre-established regions; in general, the line that is chosen still satisfies the condition even when the regions are narrowed within certain limits. The precession of Mercury's perihelion, however, to name one example, was from the beginning a reminder of the limits of accuracy of Newton's law.

But mathematical accuracy cannot always be equated with correctness, as Wigner himself recognizes.[8] Let's consider an example. The Babylonians and other peoples had from the earliest antiquity amassed an impressive quantity of numerical data related to the motion of heavenly bodies. These data concerned movement and relative position. Many *different* mathematical models, within different conceptual schemata, were offered throughout history to make mathematical sense of them. Although differing in many aspects, d*ifferent* models, Ptolemy's (geocentric), Copernicus' (heliocentric) or Tycho Brahe's (mix of heliocentric and geocentric) were *sufficiently accurate* to both make sense of the data *and allow for good previsions*. Ptolemy's model had to resort to things like epicycles, deferents, "excentric" points and equant points to concoct a picture of reality that *accounted for* the "observable data", *but did it with a sufficient degree of accuracy*. From our perspective Ptolemy's is a *wrong* picture of reality, but it is nonetheless good enough to save the phenomena. In fact, *any* amount of astronomical data concerning the movement of bodies in the solar system can be "saved" to *arbitrary degree of accuracy* within Ptolemy's system by using enough epicycles.

Norwood Russell Hanson says:[9]

> There is no bilaterally-symmetrical, nor excentrically-periodic curve used in any branch of astrophysics or observational astronomy today which could not be smoothly plotted as the resultant motion of a point turning within a constellation of epicycles, finite in numbers, revolving upon a fixed deferent

Hanson's paper concludes with the following remark: "[…] Ptolemy's mathematics was, in principle, as powerful, at least for the special problems before him, as is our own in dealing with the same problems". So, the accuracy of mathematical theories vis-à-vis either the data available at the moment of their elaboration *or data which become available later*, *no matter to which degree*, cannot count as an indication of the ability of these theories to "miraculously" describe empirical *reality* accurately.

Wigner is also puzzled by the fact that Newton's law involves highly non-intuitive mathematical concepts such as that of a second derivative. The fact, however, is that this notion does not have anything to do with our intuitive, perceptual experience of nature, only with a mathematical surrogate of it. Mathematical concepts do not apply to nature as *perceived*, only to nature as intentionally *conceived*, that is, physical nature, an abstract, idealized, and often mathematically enriched version of the former.

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[8] Wigner provides some examples in his paper, in particular the free-electron theory. He says, referring to physical theories, that "their accuracy may not prove their truth and consistency".
[9] "The Mathematical Power of Epicyclical Astronomy", *Isis* **51**(2): 150-158, 1960, p.154-5

Another feature of physical laws that impresses Wigner is the happy association of limited scope with accuracy. Mathematics, he believes, has the power to isolate particular aspects of reality and then describe them with astonishing accuracy. I have already showed that accuracy is not an indication of truth, in some sense of truth adequate for mathematical laws of nature (which obviously involves more than mere adequacy to the data). Now, as shown above, the constitution of physical nature involves *selecting* from perceptual nature precisely those features that can be mathematically represented. Mathematical laws of nature, which express mathematical relations among mathematical idealizations and mathematical representatives of specific features of experience, have the scope we give them. The surprising fact is that these laws can sometimes be extended to domains other than those in which they were established*, even if prima facie they do not even make sense in these new domains*. Wigner provides examples of this phenomenon. First, the quantum rules devised empirically by Heisenberg were given a mathematical formulation in terms of matrices by Born and others. The mathematical theory was then applied to problems other than those that originated the empirical calculus in the first place (even to problems for which this calculus was meaningless), proving to be in fine agreement with the empirical data. Second, the theory of the Lamb shift, a mathematical theory with practically no empirical support, which, nonetheless, proves to be extremely accurate in confront with experience. In both cases, according to Wigner, one gets out of the mathematical theory *more* than one has put in, and this, he thinks, defies understanding and explanation. I beg to differ.

The most relevant fact to be noticed is that mathematical laws can only express formal-structural properties of their domains, which, remember, are *mathematical* domains where *certain* features of experience can be represented, but only in *idealized form*. Mathematical laws are formal laws. As Hermann Weyl put so candidly, mathematics can only express what is most *superficial* in experience, its formal properties, *which different domains of experience can share*. It is a trivial fact that even radically different contexts can have the *same* or *similar* formal properties. So, there is no a priori reason why *particular* mathematical domains cannot (1) represent *other aspects* of the *same realm of* experience for which they provides adequate representation or, (2) be formally similar to *other mathematical domains*, representing *different realms of experience*. The fact that laws of physical nature, adequate for particular realms of experience, happen to be adequate for different realms of experience, which may not have anything to do with the original ones, is a trivial consequence of the fact that the laws of physical nature are formal-structural laws, which different domains can well share. Hence, the fact that a *particular* law of physical nature can be extended to realms of experience other than those for which it was designed should not be a source of surprise, for any such law can in principle,

although not necessarily be so extended. To get more from mathematics than what was put into it is a happy, but explainable and understandable fact.

There is a third, and to some more puzzling aspect of the applicability of mathematics in science, the heuristic uses it can be put to. Let's consider a paradigmatic case, Maxwell's supposed "discovery" of displacement currents and electromagnetic waves, discussed by Marc Steiner.[10] Steiner is a philosopher who turned Wigner's puzzlement concerning the effectiveness of mathematics in science into a challenge to *naturalism*, the view, he explains, that man does not have a privileged position in the natural scheme of things. For him, the effectiveness of mathematics, especially as a heuristic instrument in science, defies a naturalist conception of nature and man's place in it. This is Steiner's account of Maxwell's discovery of the concept of displacement current:[11] noticing that the original law of Ampere[12] contradicted the equation of continuity, which expresses the conservation of electric charge[13], Maxwell saw fit to add another term to Ampere's law, call it $\mathbf{X}$[14]. Now, the equation of continuity implies that the divergence of $\mathbf{X}$ is equal to the time derivative of the volume density of electric charge[15]; it then follows from Poisson's equation[16] that $\mathbf{X}$ is proportional to the time derivative of the electric field[17]. So, if this term is allowed to join the density of "real" (i.e. conduction) electric flow in Ampere's law, electric charge is conserved. Ampere's law now has two terms, one due to the conduction and another to the *displacement current*[18]. The additional term, moreover, as Maxwell clearly saw, implies the existence of electromagnetic waves (later shown by Hertz in fact to exist).

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[10] "The Application of Mathematics to Natural Science", *The Journal of Philosophy*, 86 (9): 449-480.

[11] In Steiner's own words (op. cit. p, 458): "Maxwell's procedure in writing down his immortal equations provides another example of this strategy. Once the phenomenological laws of Faraday, Coulomb, and Ampere had been given differential form, Maxwell noted that they contradict the conservation of electrical charge, though the phenomenological laws were strictly in accord with the evidence then available. Yet, by tinkering with Ampere's law, adding to it the 'displacement current', Maxwell succeeded in getting the laws actually to imply charge conservation. With no other empirical warrant (Ampere's law stood up well experimentally; on the other hand, there was 'very little experimental evidence' for the physical existence of a 'displacement current' [*these words are Maxwell's own; he however is not referring to the existence of a displacement current, but to the fact that, like "real", conduction currents, displacement currents can also produce magnetic effects JdS*]), Maxwell changed Ampere's law to read that (the 'curl' of) the magnetic field is given by the sum of the 'real' current and the 'displacement current'. Ignoring the empirical basis for Ampere's law (magnetism is caused by an electric current), but by formal mathematical analogy, Maxwell now asserted the law even for a zero 'real' current! Thus did Maxwell predict electromagnetic radiation, produced later by Hertz".

[12] curl $\mathbf{H} = 4\pi/c\ \mathbf{i}$, where $\mathbf{i}$ is the density of electric flow, i.e. the amount of electric charge crossing per unit time a unit surface perpendicular to the flow, from which it follows that div $\mathbf{i} = 0$.

[13] div $\mathbf{i} + \partial\rho/\partial t$, ρ = volume charge density.

[14] The law now reads: curl $\mathbf{H} = 4\pi/c\ (\mathbf{i} + \mathbf{X})$.

[15] div $\mathbf{X} = \partial\rho/\partial t$

[16] div $\mathbf{E} = 4\pi\rho$, and so div $\partial\mathbf{E}/\partial t = 4\pi\partial\rho/\partial t$.

[17] $\mathbf{X} = 1/4\pi\partial\mathbf{E}/\partial t$.

[18] curl $\mathbf{H} = 4\pi/c\ \mathbf{i} + 1/c\ \partial\mathbf{E}/\partial t$, $1/c(\partial\mathbf{E}/\partial t)$ being the displacement current.

This line of reasoning, Seiner claims, is purely formal; supposedly, there was no *physical* or *empirical* reason for the introduction of the displacement current. Therefore, he concludes, Maxwell's heuristic methodology relied solely on second-order mathematical analogies. However, even if this were a historically faithful account (which it isn't) Steiner's conclusions would not be warranted, for if this were how Maxwell hit on the notion of displacement current he would have had a *physical* motivation for it, namely, *that electric charge must be conserved*. Mathematical manipulations would have been used only to find out the *form* – and *only* the form – the flow of "missing" charge took.[19] They would have shown him that the density of this flow *had to be formally equivalent* to the time derivative of the electric field, *which could then be seen as formally equivalent to the density of an electric current*, the displacement current, *even in the absence of conduction currents*.

The above, however, is an unfaithful account of historical facts; the notion of displacement current was in fact *naturally required* by Maxwell *physical* model of electromagnetic phenomena; if anything, his discovery was based on *formal* analogies between *physically different domains of experience*. Electromagnetic action has, of course, mechanical effects. This naturally led Maxwell to conceive of a mechanism – in the literal sense of the term – that would produce these same effects; that is, to the elaboration of a mechanical model of electromagnetic phenomena (see Maxwell's "On Physical Lines of Force", 1861-2). The basic requirement was, of course, that the model had to behave in a formally equivalent manner to electromagnetic action with respect to its mechanical effects. In this model magnetic flux, in both conductor and insulators (*including the vacuum*), is directed along rotating vortex tubes; space (including *empty* space, it is important to emphasize, which was not seen as empty at all, but filled with ether) was, Maxwell imagined, filled with these rotating tubes of magnetic flux, between which he inserted "idle wheels" in order to eliminate friction. He identified the movement of these wheels with electric current. In insulators, they wouldn't be able to move freely, but could oscillate around their position of equilibrium. Maxwell then used this model to account also for the storage of electrical energy in insulators, which, of course, had to be done in some mechanical form. He supposed that in dielectrics the electric particles, when placed under the action of an electric field, would be forced to move from their equilibrium position – they would be *displaced* – thus storing potential mechanical energy.

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[19] This is a way mathematics can typically play a heuristic role in science, namely, by rendering explicit the formal consequences of certain general principles, such as the conservation of electric charge, in this case, or the principle of conservation of energy, as Michel Patty (La matière dérobée. L'appropriation critique de l'objet de la physique contemporaine, Paris: Éditions des Archives Contemporaines, 1989, particularly chapter IX, entitled "Modèles mathématiques et réalité physique, pp. 319-57) claims was the case in the "discovery" of the neutrino by Pauli.

Time-dependent electric fields would then give origin to small displacements appearing as small electric currents that would propagate as an electric current through the medium. Displacement current had been discovered. Since the displacement is proportional to the electric field **E**, the displacement current, i.e. the time derivative of the displacement is proportional to the time derivative of **E**. The density of displacement current can then be written as a multiple of $\partial\mathbf{E}/\partial t$ and allowed to join the density of *conduction* current in Ampere's law. In short, for Maxwell, a variable electric field would, *even in the vacuum*, *even in the absence of conduction currents, simply because his physical model of reality would so require*, originate a variable displacement current.

But the elaborate mechanical machinery Maxwell devised is a *fiction*; it does not correspond *materially* to reality[20]. What does the fact that reality can be modeled by heuristically useful *fictions* tell us about the nature of physical theories and the applicability of mathematics? Since reality and Maxwell's model are only *formally* similar, formal similarities must figure prominently in the answer.

According to Maxwell (*Analogies in Nature*, 1856), we can investigate a realm of scientific interest by investigating another, materially different from, but *formally* identical to it, *because* science is interested in *relations* among things rather than things themselves. If phenomena we know well and phenomena we know less well (for example, mechanical behavior of fluid vortex tubes vis-à-vis mechanical behavior of electromagnetic action) have at least *some* formal properties in common it is reasonable to suppose they may have *more* formal properties in common. It is then a good idea to investigate the matter; it may pay off or it may not. If formal identity extends *further* or *deeper* than what was at first apparent (as we are justified to expect) it probably will[21].

But, since formal similarities are all that matters in the modeling of reality, physical models of the type Maxwell used play only the role of physical supports of formal structure and can be substituted, often with advantage, by mathematical models.[22] So, purely mathematical manipulations within mathematical models of some conveniently idealized aspects of

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[20] Maxwell is quite clear about the purely formal relevance of his mechanical model: "the conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connection existing in Nature […]. It is however a mode of connection which is mechanically conceivable and it serves to bring out the actual mechanical connection between known electromagnetic phenomena." (Scientific Papers, vol. 1: 486, apud M. Longair *Theoretical Concepts in Physics: An alternative view of theoretical reasoning in physics*, Cambridge: CUP, 2003: 88-98, p. 97)

[21] Maxwell's model was admirably successful in accounting for all electromagnetic phenomena known at the time.

[22] Physical models (like Maxwell's) are superior only with respect to visualization.

experience can display unexpected heuristic virtues, not by directly pointing to hitherto unknown possibilities of experience, but by revealing particularly interesting formal properties of the models, whose material bases *may* consist in hitherto unknown possibilities of experience. It must be emphasized, however, that the connection between formal properties of mathematical models of reality and *particular* material properties of perceptual nature can only be verified a posteriori, by inquiring nature directly in experience, never by simply playing with the mathematical formalism, exclusively concerned as this is with properties for which there are always different possibilities of material instantiation.

It is interesting to see what Maxwell himself has to say about his methodology. He says in the essay *Analogies in Nature*:[23]

> Whenever [men] see a relation between two things they know well, and think they see there must be a similar relation between things less known, they reason from one to the other. This supposes that, although pairs of things may differ widely from each other, the *relation* in the one pair may be the same as that in the other. Now, as in scientific point of view the *relation* is the most important thing to know, a knowledge of the one thing leads us a long way towards knowledge of the other.

Maxwell had, then, a perfectly simple explanation for the success of formal analogies in scientific heuristics. Models, whether physical or mathematical, model because they are indistinguishable from reality with respect to *form* (or underlying structure; Maxwell's "relations"), at least as far as the model "works". Now, if a model shares with perceptual reality, conveniently idealized, a core of common formal properties, it is reasonable to explore the model for hints of further formal properties of reality. The model can behave correctly beyond the limits where it has already been proved formally correct. Maxwell's successful mechanical model of electromagnetic phenomena told him that variable electric fields generate displacement currents; if these currents had the same magnetic effects of conduction currents, electric charge would be conserved. Since electric charges *must* be conserved, the incorporation of displacement current in Ampere's law is *physically* justified (which does *not* mean that further *direct* empirical evidence would no longer be required.

We can summarize Maxwell's strategy thus: realms of scientific interest can be investigated through others, materially different from but *formally* identical in whole or in part with them because formal properties can be *identical* even when the things displaying these properties are materially different, and if phenomena we know well display *some* formal properties that are identical with formal properties of phenomena we know less well (for example, mechanical behavior of fluid vortex tubes vis-à-vis mechanical behavior of

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[23] Published in 1856, apud M. Longair 2003, p.88.

electromagnetic flux) it is a good idea to explore this formal identity for heuristic purposes. It may pay off or it may not; if the identity extends *further* than already observed (as we are justified to expect) it probably will.

It is now obvious how this strategy can be extended so as to allow mathematics to play a relevant heuristic role in science. Since the utility of mechanical models is solely due to formal similarities, material content being irrelevant, we can use instead *mathematical* models. It is then *possible*, but only *possible*, that *formal* properties of perceptual nature reveal themselves in the mathematical formalism devised for the investigation of physical nature before they show up in perception. The formalism, however, particularly if it involves terms that are not given a priori meaning, *cannot* by itself determine either whether formal possibilities c a n b e e f f e c t i v e l y m a t e r i a l i z e d perceptually or, in case they are, in which *particular* states-of-things.

I think that I can already conclude. My point is that the supposed mystery involved in the applicability of mathematics in physics, in any shape or form it takes, particularly as a heuristic instrument, which puzzled Wigner and Steiner, is in fact a pseudo-problem rooted in philosophical *partis-pris*. These authors take a *realist* stand with respect to the question that prejudges the issue. They confound physical nature, a somewhat simplified and idealized representation of certain aspects of perceptual nature, those precisely that can be given mathematical expression, devised for methodological purposes, with Nature herself, taking for a *given* what is only an *intentional construct*.

Perceptual nature has, of course, already, to some extent, a mathematical structure, simply because we perceive Nature as a structured system and mathematics deals with structures, be they materially instantiated in given systems or emptily conceived in abstract. To suppose that the structures perceived in empirical experience belong to Nature out there, existing independently of perception, is a philosophical thesis we have no reason to accept uncritically. But even if they are indeed independent of the action of perceptual systems, perceptual structures are usually mathematically too poor to invite a more sophisticated mathematical approach; in general only by the action of idealization they become fully mathematical.

By idealization perceptual nature is turned into physical nature, a mathematical manifold, which obviously invites a mathematical approach, the raison-d'être of the

whole process. Mathematics is intrinsic to physical nature simply because physical nature is tailor-made for mathematics. As any mathematical manifold, physical nature can be further enriched mathematically, inviting ever more sophisticated mathematical treatment. This alone accounts for the heuristic efficacy of mathematics in physics, in no measure different from the efficacy of mathematics in mathematics itself. But if in mathematics mathematical facts stand no matter what, in science they require being interpreted as phenomena in perceptual nature and empirically verified. No mathematical indication of hitherto unknown phenomena –which invariably occur at a purely formal-structural level only – can count as an outright "discovery" of anything in perceptual nature independently of, first, being given a material content, either in a previously determined context or a new one, and being effectively perceived in the context where its mathematical indicator is interpreted. Material content, as we saw, is never predetermined theoretically, and so scientific discoveries can *never* be only a matter of mathematical manipulations.