Mathematics for Quantum Gravity

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Outline

Introduction

"Quantum gravity has become the holy grail of theoretical physics."

Physical Systems

A Physical Systems is represented by a family of mathematical objects
(is that a *definition*, or a *characterization* of the system ?)
(ontological question: ; see Federico Zalamea)
Example : A [classical, non-relativistic] dynamical system:,
a symplectic manifold (phase space) with a function on it (Hamiltonian)
Example : A given quantum [non-relativistic] system is defined by a an algebra of operators, including the Hamiltonian, and acting on an Hilbert space.

Quantization: going from one description to the other

Physical Systems

A physical system is characterized (defined ?) (at least) by
- a *set of states* in which it can be
- and a set of *observables*, which characterize the specific measurements that can be performed on the system.

(Observables may depend on the observer !)

A state may be defined as the collection of all the possible outcomes of all possible measurements.

$\rightarrow$ To express a theory (classical or quantum, relativistic to not):
- identify the states and observables.
- recognize the physical (= real) state among the possible states (equations).
- find evolution.

Evolution

Two (very) different views

1) Non relativistic : evolution w.r.t. time.

At each instant of time, the system is in a given (non relativistic) state.

Evolution = a sequence of transformations parametrized by time: $s(t_0) \to s(t)$:

a one parameter group of transformations on the states: $s(t) = u(t)\ s(t_0)$:

a (temporal) flow (unitary in quantum physics)

2) relativistic: there is no time to monitor evolution.

(a possible solution: chose an arbitrary parameter (*time function*) which allow to treat the system like NR).

or covariant view: a *relativistic state* is the complete evolution of the system; see examples

Evolution: Examples

Example: the particle:
NR view : a position in space; which evolves with time
relativistic view: a trajectory in space-time= relativistic state
Example: the scalar field:
NR view : a function of space at each value of time
relativistic view: a function on space-time= relativistic state
Example: gravitation:
NR view = geometrodynamics : a metric $h(t)$ on each spatial section, evolving w.r.t. a time function;
relativistic view: a metric g on space-time= relativistic state
conflict ?

Quantum Physics: States and Observables

Observable acts on state: to give
- the possible outcomes (real numbers)
- a probability distribution for each outcomes $\rightarrow$ an expectation value

Linear superposition of states, evolution, measure, reduction , collapse...

Hilbert space view and *algebraic* view are dual.

Other approaches: information (state = maximal info about a system), topos (Isham), QFT as a functor (Atiyah): geometrical category $\rightarrow$ algebraic category

Hilbert space view

To a quantum system is associated an Hilbert space
pure state = ray = projector P
general state = convex lin combinations of pure states: density matrix $\rho = \sum_a a\ P_a$
observable = Hermitian (self-adjoint) operators on H
spectrum = set of eigenvalues (=possible outcomes) ;
spectral decomposition $A = \sum_a a\ P_a$
(Pa = orthogonal projector onto eigenspace associated to a).
Unitary evolution and collapse (= reduction = measurement)

Algebraic view

To a quantum system is associated an an unital $C^\star$-algebra.

Observables are viewed as abstract self-adjoint objects.
The *(algebraic) states* appear as a derived concept: linear forms acting on $\mathcal{A}$.
The algebraic version of a pure Hilbert state $\psi =$ linear form

$$\omega_\psi : \; A \to \langle A\psi \mid \psi \rangle$$

acting on an operator A as *expectation value*.
alg. version of a *density operator* ρ ($=$ general Hilbert state):

$$\omega_\rho : \; A \mapsto \omega_\rho(A) = \mathrm{Tr}(A\rho).$$

Gelfand Naimark Segal (GNS) construction of Hilbert space.

Remark on States

Classical States: (in classical Hamiltonian mechanics)
pure state= point of the phase space Γ;
general state = pdf on the phase space Γ; (pure: distribution)
Observable = element of the commutative
C*-algebra $A = C(\Gamma)$ of smooth complex functions.
Gelfand-Neimark: the state space is the G–spectrum of $C(\Gamma)$
Category equivalence : commutative C*-algebras – compact spaces
generalizion to nc case $\rightarrow$ non-commutative geometry.
quantum physics = non-commutative phase space

Evolution

Classical (non relativistic) Evolution:
hamiltonian (symplectic) flow in Γ = one param. group of automorphism (symplectomorphisms) in A.
relativistic state = orbit of the flow.

Quantum (non relat.) Evolution
Unitary flow in state space.
relat. state = orbit, or *constant state in Heisenberg picture*.

Classical gravity : gr

Algebra $\rightarrow$ Geometry

Riemanian Geometry

Independent structures in a *differentiable manifold*
(= topological manifold with differential structure)

- *metrics* : symmetric tensors of type 2,0 (10 dofs);
  $\rightarrow$ *[pseudo–]Riemanian manifold* (M,g)
- *connections* = holonomy = cov. derivative =... (64 dofs)
  each characterized by torsion, curvature, (40 symmetric)

Non-metricity ∇g. A metric manifold admits 24 metric connections. Only one symmetric = Levi-Civita connection. used in general relativity (parallel transport)
To g is associated a unique Levi-Civita connection ∇
Curvature of ∇ = Rieman tensor $\rightarrow$ Ricci, Einstein, scalar,...
These properties of ∇ are associated to g via ∇.

Tetrads and cotetrads

An other way to express the metric!

Given a metric g, *tetrad* (= ON frame)
is a set of 4 vectors $e = (e_I)$ verifying $g(e_I, e_J) = \eta_{IJ}$.
A *cotetrad* (= ON coframe) is a set of 4 one-forms $\theta = (\theta^I)$ verifying $g(\theta^I, \theta^J) = \eta^{IJ}$.
(the reciprocal: $\langle \theta^I, e_J \rangle = \delta^I_J$).
Conversely, a cotetrad θ defines the metric $g = \eta_{IJ}\ \theta^I \otimes \theta^J$.
Cotetrads related by LLTs define the same metric.
one metric $\equiv$ one family of cotetrads related by (local) Lorentz transforms.
one metric $\equiv$ a cotetrad modulo (local) Lorentz group.

gr as classical field theory

find the metric g on a given differentiable manifold M (given limiting conditions, given source of gravity)
source = matter-energy energy-momentum tensor T Einstein equation: Einst $= ct$

Four options (at least) for dynamical variables of GR :
- g alone (Einstein–Hilbert);
- g and connection (assumed torsionless) (Palatini)
- cotetrad θ (modulo Lorentz) (Dirac)
- cotetrad θ (modulo Lorentz) and connection (assumed metric)

Also Covariant or canonical treatment.

Problems for quantum gravity

- Diffeomorphism invariance
- problem of time; no Hamiltonian or symplectic structure
- No metric, no causlity
- No measure, no Hilbert space
- Observer ?
- One Universe : probabilistic interpretation ?

Loop Quantum Gravity (LQG)

Three preliminary steps

- 3+1 formalism: $\rightarrow$ Geometrodynamics
split space-time as *space evolving w.r.t. time [function] T*;
Evolution of a *Riemanian* structure h on a fixed 3 manifold Σ.
- use metric g *and connection* ∇ (rather than metric alone)
as variable: connecto–dynamics
- replace metric by cotetrad: $g \rightsquigarrow \theta$ (with *Gauss constraint*).

$\rightarrow$ **new variables** :
- lapse and shift indicate the shape of Σ in M: unphysical, not dynamical $\rightarrow$ two constraints
- cotriad e and connection form ω, on Σ, evolving with T.

LQG: a canonical system

$\rightarrow$ **canonical formalism !** (symplectic)
Hamiltonian = sum of constraints (relativistic systems)
canonical variables: E ("electric field") and ω.

$\rightarrow$ **Analogy with gauge theories**: ω is a G– connection form ;
$G = SO(3) \simeq SU(2)$, or Lorentz $\simeq \mathrm{SL}(2, \mathbb{C})$

Topological theory with constraints (Plebansky formalism, Holst action)

Loose covariance
Difficult to quantize because of constraints

New variables

Difficult to quantize because of constraints $\rightarrow$
New variables (canonical transformation)
involving a *Barbero–Immirizi parameter*
Original case $\beta = i$ (*Ashtekar*) ; $\rightarrow$ problems, but ... (Noui)
$\beta \in \mathbb{R}$: $\rightarrow$ LQG theory.

$$(\omega, E) \rightsquigarrow (A_\beta, E)$$

Poisson brackets $\{A, E\}_{\mathrm{P}} = \delta$.

Quantization?

Usual quantization (quantum mechanics) :
SYMPLECTIC $\rightarrow$ UNITARY
commutative $\rightarrow$ non commutative
Configuration space $= C = \{q\}$.
Phase space $\Gamma \ni (q, p)$.
quantization : $\Gamma \rightarrow$ Hilbert space
wave functions $\psi(q) =$ function $C \rightarrow \mathbb{C}$
Hilbert space $L^2(\mathrm{C})$: square –integrable functions
(requires a *measure* on C) !!!

LQG: wave functions $\psi(A)$?
(unconstrained) config. space $= \mathcal{A}=$ space of
connections–form on the differentiable manifold Σ.
wave function $\psi(A)$? : $\mathcal{A} \rightarrow \mathbb{C}$; through holonomies
Hilbert space ?

Holonomies

A G–connection is represented by a $\underline{\mathfrak{g}}$-valued connection form A
Integration of one-form, A along a path $c \to \int_c A \in \underline{\mathfrak{g}}$.
Since $\exp(\underline{\mathfrak{g}}) = G$,

$$hol_c(A) \stackrel{\text{def}}{=} \exp(\int_c A \in \underline{\mathfrak{g}}) \in G,$$

the holonomy of A along the path c: a function $\mathcal{A} \to G$.

Compose with a function $G \to \mathbb{C}$ to obtain a function $\mathcal{A} \to \mathbb{C}$.
Use basic functions (e.g., spherical harmonics) indexed by a *spin index* j
To each (c, j) corresponds a basic wave function $\mathcal{A} \to \mathbb{C}$.

Wave functions

To each (c, j) corresponds a basic wave function $\mathcal{A} \to \mathbb{C}$. To form an Hilbert space, combine indexed paths (c, j) to form spin networks (Rovelli, Smolin).

A *spin network* is a directed graph colored with spins:
each edge e is labelled by a unitary irreducible representation of a [gauge] group,
thus a half integer number for the group SO(3) or SU(2);
each vertex carries an intertwiner in the tensor product.
Spin networks have been first developed by Roger Penrose, and then rediscovered as the result of canonical quantization of GR in Ashtekar variables. (lqg)

Spin networks

Spin networks are embedded in Σ.
Abstract spin networks (not embedded) are wave functions which solve the diffeomorphism constraint (Rovelli, Smolin) !.
An Hilbert space ?
define a inner product (complicate see below):
Spin networks form an ON basis for that Hilbert space (however non separable).
This requires to enlarge the configuration space $\mathcal{A} \rightsquigarrow \overline{\mathcal{A}}$ generalized connections.

Groupoids

A groupoid is a category with all morphisms invertible.
Groupoids form the [cartesian closed] category **Gpd** .
Groupoid morphisms = functors in **Gpd** .

A *group* G= a groupoid $\widehat{G}$ with one object.
Gpd contains the category of groups as a full subcategory.

The *Path groupoid* PM of a manifold M is the first and simplest way of regarding it as a category:
the objects are the points of M; the arrows the oriented [smooth holonomy classes of] paths between ??points.

Connections as holonomy functors

A local connection form A on M is a groupoid morphism: smooth functor in **Gpd** :
path groupoid $PM \rightarrow$ gauge group $\widehat{G}$ (its groupoid)
$PM \rightarrow \widehat{G} : c \rightarrow hol_A : \; c \rightarrow hol_c(A) = hol_A(c) \in G,$
$(=$ *holonomy functor* $=$ *holonomy map*$)$

Space of connection forms $\mathcal{A}$ is a category
There is a categorical equivalence

$$\boxed{\mathcal{A} \sim SmoothFunct(\mathcal{P}_M, BG) = SmoothMor(\mathcal{P}_M, \widehat{G})}$$

gauge transformation $=$ natural isomorphism between functors.
($\mathcal{A}$ is a subcat of $\overline{\mathcal{A}} \stackrel{\text{def}}{=} Mor(\mathcal{P}_M, \widehat{G})$ (non smooth): a distributional extension of $\mathcal{A}$. An object is a *generalized connection* (connection when smooth).

Graphs

(graph mean oriented graph.)

- See a graph Γ as a [path] groupoid P_Γ: objects are vertices; arrows are edges. P_Γ is an object in **Gpd** .
- Inclusion of graphs $\Rightarrow$ inclusion of graph-groupoids : groupoid morphism $i: \mathcal{P}_\Gamma \subset \mathcal{P}'_\Gamma$
- *embedded graph* in M= graph Γ with an *inclusion functor* (morphism in **Gpd**): $F: P_\Gamma \to \mathcal{P}(M)$.
- Embedded graphs form the cat $\mathcal{L}$: an *inductive family* in the cat **Gpd** ; admits $\mathcal{P}(M)$ as colimit (*inductive limit*) when Γ describes all embedded graphs.
- A connection form A in M has a pull-back in a graph Γ: the groupoid morphism

$$A_\Gamma: P_\Gamma \to \hat{G}.$$

(the restriction of A to Γ : a connection-form on Γ). Their space $\mathcal{A}_\Gamma = \{A_\Gamma\}$ is the restriction of $\mathcal{A}$ to Γ.

Graphs

- For each Γ, $\mathcal{A}_\Gamma$ is isomorphic to $\prod \mathcal{A}_e = G^{E(\Gamma)}$.
- The compact Lie group G admits the Haar measure $\mathrm{d}\mu \Rightarrow$ $\mathcal{A}_\Gamma \simeq G^E$ admits the product measure $\mathrm{d}\mu_\Gamma = (\mathrm{d}\mu)^E$.
- The $\mathcal{A}_\Gamma$ admit the *projective limit* $\overline{\mathcal{A}}$, the space of *generalized connections* (discontinuous, but they assign an holonomy to each curve.). It acquires naturally the inductive limit of Haar measures: the *Ashtekar-Lewandowski measure*.
- $\mathcal{A}$ (the space of smooth connections) is densely embedded in $\overline{\mathcal{A}}$ (a completion of $\mathcal{A}$)

Functions and Hilbert space

- For each graph, $\mathcal{C}(\mathcal{A}_\Gamma)$= {continuous functions of the A_Γ};
- Haar measure $\rightarrow$ Hilbert space $L^2(\mathcal{A}_\Gamma, \mathrm{d}\mu_\Gamma) \simeq L^2(G^{(\Gamma)})$.
- inductive limit of $L^2(\overline{\mathcal{A}}_\Gamma)$: $H_{kin} = L^2(\overline{\mathcal{A}})$.
- The AL measure defines a scalar product

$$\langle \varphi, \psi \rangle = \int \mathrm{d}\mu_{AL}\ (A)\ \varphi^*(A)\ \psi(A).$$

- kinematical Hilbert space H_{kin} ofLQG.
  It is non-separable.
  It admits a (ON) basis of spin network .
- Basic multiplication and derivation operators in H_{kin}= quantized holonomy and flux operators hL and F^S.
  One can impose the constraints as operator constraints.

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