Constructive Axiomatic Method and Modern Physics

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D. Hilbert Foundations of Mathematics (1927):

The Axiomatic Method is to become "the basic instrument of all research" ():
A formal axiomatic presentation of mathematical and scientific theories should become a standard routine in the future mathematical and scientific practice. The primary purpose of axiomatic presentation is the logical justification of scientific theories.

Hilbert 1900, number 6 of 23:
"Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part."

Mathematics is to be axiomatized first; other sciences will follow the example of mathematics along with the further mathematization of these sciences.

Cf. Kant: "I assert that in any special doctrine of nature there can be only as much proper science as there is mathematics therein." Metaphysical Foundations of Natural Sciences (1786)

WHERE WE ARE NOW?

after the past 100 years of continuing efforts towards the axiomatization

Mathematics

- A (meta-)mathematical study of formal axiomatic theories of mathematics (like ZF or Peano arithmetics) is now a well-established field of research called (vaguely) the Foundations of Mathematics (FOM). It comprises parts of Mathematical Logic, Model Theory and the whole of modern Set Theory.

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- The record of FOM includes important mathematical achievements such as the proof of independence of CH from ZF (Gödel-Cohen). In FOM people use <u>informal</u> (and only <u>informal</u>) reasoning (just as elsewhere in mathematics as we shall see). In FOM formal theories are *objects* of study; theoretical components of FOM itself (such as Cohen's argument based on the method of forcing) are <u>not</u> formal; theories of FOM are not axiomatic.

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- Nevertheless it is claimed that all (or almost all) well-standing mathematical theories allow for a standard formal axiomatic presentation (aka formalization) *in principle* on the basis of ZF. This claim is supported by informal arguments (i.e., proof) similar to those used elsewhere in mathematics. So epistemically it should be treated on equal footing with all other usual mathematical claims (theorems). The "standard formal axiomatic presentation" of given informal theory T is an abstract mathematical object similar to objects treated in T (be it an algebraic structure, a space or whatnot). The popular use of the term "presentation" in this context is questionable.

An *informal* version of Axiomatic Method is used in mathematics more widely than its formal counterpart. It can be seen as a result of practical compromise between the logical rigor and the traditional mathematical practice. An important example of such a practical compromise is Bourbaki's *Eléments de mathématique*. Bourbaki mathematics is a Tarski-style *model theory* of semi-formal axiomatic theories such as Group Theory. Here Set Theory is used for building models of other theories. Such models are conventionally called *mathematical structures*.

I claim that the *constructive* version of formal axiomatic method described below makes more justice to the informal axiomatic method than the formal axiomatic method in its standard form.

Conclusion on Mathematics:

The standard Hilbert-style formal axiomatic method serves us as a tool for studying a logical structure of mathematical theories but not as a tool for presentation and logical justification of these theories (as Hilbert and his followers wanted). The requirement according to which a given theory T must be formalizable *in principle* constitutes a very weak necessary (but not sufficient) condition for saying that T well-standing. Notice that a contradictory theory can be well formalizable.

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- "The axiomatization of physicists": von Neumann 1932 *Mathematische Grundlagen der Quantenmechanik*. Quantum Logic as the logical foundation of Quantum Physics.

Bunge on v. Neumann

In his epoch-making book, which enriched the mathematical framework of the theory, von Neumann is *wrongly* supposed to have laid down the axiomatic foundations of quantum mechanics. As a matter of fact his exposition lacks all the characteristics of modern axiomatics: it does not disclose the presuppositions, it does not identify the basic concepts of the theory, it does not list all the initial assumptions (axioms), it fails to propose a consistent physical interpretation of the formalism, and it is ridden with inconsistencies and philosophical naivetés. Yet for some strange reason it passes for a model of physical axiomatics.

WHAT IS WRONG?

Bunge after Tarski

There is a single theory that starts from scratch: mathematical logic [..] All other theories presuppose at least logic and usually a lot more. More precisely, the least a mathematical or a scientific theory takes for granted is the so-called ordinary (two-valued) predicate calculus enriched with the microtheory of identity. This theory is necessary and sufficient to analyse the concepts, formulas, and reasonings occuring in mathematics and in science - or rather to analyse their form.

Suppes and his followers certainly buy this. Von Neumann does not: recall his idea of *quantum logic*. Debate Putnam vs. Dammit.

Tarski 1957

"A possible realization in which all valid sentences of a theory T are satisfied is called a model of T."

A usual assumption: only non-logical terms are multiply interpreted while the logical semantics is rigidly fixed. This is the sense in which Logic has a fundamental status in this framework. This feature is absent from the constructive variety of the notion of axiomatic theory, which we are now going to discuss.

Constructive Axiomatic Method

The received notion of axiomatic theory is that of a set of *propositions* (fully interpreted or not) provided with a relation of deducibility.

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- Warning: My use of the term "constructive" is less restrictive than usual. However it is not new and found, in particular in Hilbert&Bernays 1934 and Kolmogorov 1932.

Slogan 1

Rules are more fundamental than axioms.

This accords with the standard Hilbert-style axiomatic approach if by rules one means logical rules. The standard approach conceives of logical rules as rules of handling propositions (ex. *modus ponens*).

Slogan 2

Rules do not reduce to rules for handling propositions. Other types of theoretical objects should be equally handled according to certain rules.

I leave it open whether or not only rules for handling propositions qualify as *logical*. The standard axiomatic approach suggests the answer in positive. But one may wish to understand the scope of logic differently following Kant and some other influential philosophers.

Claim:

The constructive axiomatic method provides a stronger formal support for the long-established practice of **model-building** in science than does the received method.

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- In today's mathematics the notion of constructive axiomatic theory is best represented by the Homotopy Type Theory (HoTT). However there are older examples of constructive theories (in the relevant sense of "constructive") including the geometric theory of Euclid's *Elements*.

Why the constructive version of the axiomatic method is any better?

Specific models (in the Pure Mathematics and moreover in Physics and other natural sciences) do not come from the air. They do not emerge from an immediate experience, the intuition or the lucky guess. They are built from simple elements according to fixed rules. Think constructions by the ruler and compass in Euclid. Think of a Classical Newtonian model of some given mechanical setup. The standard Hilbert-style axiomatic method has no resources for accounting for this constructive aspect of theories. It uses only rules, which regulate the construction of *propositions* from simple propositions aka axioms. Only such rules are usually called the rules of logic.

Why the constructive version of the axiomatic method is any better?

The constructive method has also rules for non-propositional objects. Moreover it allows for new rules (called *derived* rules) justified on the basis of other rules.

In Mathematics the constructive approach is already proved effective: the proof of 4-colors theorem is formalized through Coq (based on MLTT, which is the logical base of HoTT). Given the prominence of model-building in Physics there is a reasonable hope that this method will also work more successfully in Physics.

Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Natural deduction

Identity: $\dfrac{}{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$ (& - intro)

$\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Correspondence

$$\& \equiv \times$$

$$\supset \equiv \rightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A\&B$ is a pair consisting of a proof of A and a proof of B

CS rule of thumb

Propositions-as-Types

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- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: extensional versus intensional

- Extensionality: Propositional identity implies definitional identity (ex. LCCC)

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- Extensionality: Propositional identity implies definitional identity (ex. LCCC)
- First intensional (albeit 1-extensional) model: Hofmann & Streicher 1994:
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

Hofmann & Streicher groupoid model

$\vdash A : type$ - groupoid A
$\vdash x : A)$ - object x of groupoid A
$Id_A(x, y) : type$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
(no reason to be trivial unless x = y!)

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

HoTT: the idea

Types in MLTT can be modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in category theory.

Homotopy model of MLTT

- In the groupoid model of MLTT groupoids are *fundamental groupoids* (i.e., groupoids of paths) of topological spaces .
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

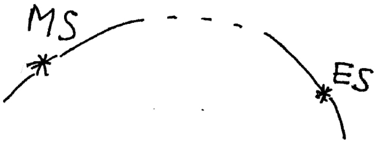
Propositional reduction as truncation

Each type is "made into" a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

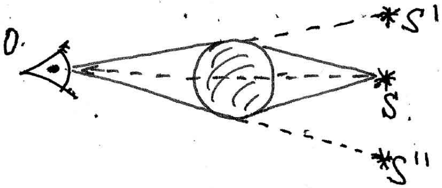
Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as constructive in the above sense (under certain conditions also in a stronger sense of being Turing-computable).

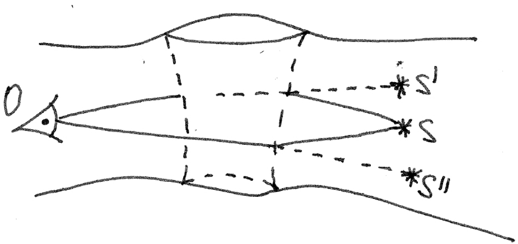
Identity through time



Gravitational lensing

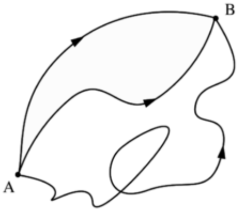


Wormhole lensing



Quantum trajectories

truncation: quantum $\longrightarrow$ classical



Conclusion

IF HoTT proves to be a full-fledged foundation of mathematics it will also serve as a mathematical foundation for the mathematized Physics. Then it will covert physical theories into constructive axiomatic theories automatically. Unlike Set Theory the Homotopy (Type) Theory has some natural physical interpretations. That makes HoTT a strong candidate to the role of basic "constructor" out of which one may build physical (interpreted) theories. Theories built with HoTT are *computable*) unless one adds to MLTT new axioms such as Univalence, which may be indeed strongly motivated by logical, geometric and perhaps also specific physical reasons.

THE END