Constructive Axiomatic Method in Euclid, Hilbert and Voevodsky

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Claim 1:

The received notion of axiomatic theory as a set of *propositions* (fully interpreted or not) provided with a relation of deducibility is not adequate to the successful practice of axiomatic thinking in mathematics and physics.

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- This claim also concerns physics (Newton 1687, von Neumann 1932).

Claim 2:

Euclid's *Elements* and HoTT instantiate the notion of *constructive* axiomatic theory, which is more general than the received notion.

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- The key difference is this: a constructive theory, generally, is not a set of propositions; it treats propositions as certain types of objects along with certain other, non-propositional, types of objects. The formal distinction between propositional and non-propositional types will be explained in what follows.

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- The key difference is this: a constructive theory, generally, is not a set of propositions; it treats propositions as certain types of objects along with certain other, non-propositional, types of objects. The formal distinction between propositional and non-propositional types will be explained in what follows.
- Warning: My use of the term "constructive" is not new and found, in particular in Hilbert&Bernays 1934 and Kolmogorov 1932. Nevertheless it significantly differs from other current uses of the same term.

Slogan 1

Rules are more fundamental than axioms.

This accords with the standard Hilbert-style axiomatic approach if by rules one means logical rules. The standard approach conceives of logical rules as rules of handling propositions (ex. *modus ponens*).

Slogan 2

Rules do not reduce to rules for handling propositions. Other types of theoretical objects should be equally handled according to certain rules.

I leave it open whether or not only rules for handling propositions qualify as *logical*. The standard axiomatic approach suggests the answer in positive. But one may wish to understand the scope of logic differently following Kant and some other influential philosophers.

Claim 3:

The constructive axiomatic method provides a stronger formal support for the long-established practice of **model-building** in science than does the received method.

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- The constructive axiomatic method is evidently better implementable on **computer** than the received method. Its modern version has been implemented through COQ, AGDA and some other software.

Claim 4 (epistemological):

Knowing how is a part of scientific and technical knowledge, which is at least as much important as propositional knowledge (aka *knowing that*). Observe that the core logical knowledge (= knowledge of logical inference) belongs to the former kind. However the scientifically relevant *knowing how* does not reduce to its logical variety.

The possibility of expressing any rule in a propositional form in a meta-language has no bearing on the above: in any system of logic propositions and rules for handling these propositions (rules of inference) must be carefully distinguished.

Euclid's Common Notions (Axioms) 1-3

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.

Euclid: Postulates 1-3

P1. To draw a straight-line from any point to any point.
P2. To produce a finite straight-line continuously in a straight-line.
P3. To draw a circle with any center and radius.

P1-3 are NOT propositions but rather *rules* for primitive operations!

operation	input	output
P1	two points	segment
P2	segment	extended segment
P3	segment	circle

Problems and Theorems

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- Problem: To construct an equilateral triangle on a given finite straight-line. (**Pr1.1**)

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- Theorem: For isosceles triangles, the angles at the base are equal to one another (**T1.5**)

Propositional reduction:

P1M: Given two different points it is always *possible* to produce a straight segment from one given point to the other given point.

P1E: Given two (different) points there *exists* a straight segment having these given points as its endpoint.

Pr1.1E: Given a finite straight-line there exists an equilateral triangle having this line as its side.

The straightforward propositional reduction of Euclid along the above lines does not bring any consistent axiomatic theory.

The non-propositional character of Postulates and Problems plays an essential role in applications. The combination of ruler and compass (I mean now real instruments rather than the corresponding ideal Platonic Forms) works as an effective analogue computer useful in a large class of space-related practical problems. In such a context it is essential *how* these instruments should work rather than what sort of *truths* they may help one to discover and justify.

The non-propositional aspect of Euclid's geometry may also play a role in a theoretical context. As a part of Newtonian Mechanics Euclidean geometry allows one to build a mathematical model of, say, the Solar planetary system. Such special models are built from certain primitive elements (like points) according to certain (mathematical) rules rather than simply chosen from a stock, where they are kept in a ready-made form.

The issue here is not that of metaphysical nature of mathematical objects but rather that of effectiveness of modeling. Mathematical models of natural phenomena need to be effectively computable whatever is their metaphysical nature. The *semantic view* rules out the possibility of replacing the computing in models by the "purely logical" computing in the appropriate propositional theories.

Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the [narrow "existential"] axiomatic method and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [..], an axiomatic theory [in the narrow sense] refers to a fixed system of things (or several such systems) [i.e. to one or several models].

Bunge 1972

[V]on Neumann [in Mathematical Foundations of Quantum Mechanics of 1932] is wrongly supposed to have laid down the axiomatic foundations of quantum mechanics. As a matter of fact his exposition lacks all the characteristics of modern axiomatics [..]. Yet for some strange reason it passes for a model of physical axiomatics.

Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Natural deduction

Identity: $\dfrac{}{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$(& - intro)

$\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Correspondence

$$\& \equiv \times$$

$$\supset \equiv \rightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A\&B$ is a pair consisting of a proof of A and a proof of B

CS rule of thumb

Propositions-as-Types

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)

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- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: extensional versus intensional

- Extensionality: Propositional identity implies definitional identity (ex. LCCC)

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- Extensionality: Propositional identity implies definitional identity (ex. LCCC)
- First intensional (albeit 1-extensional) model: Hofmann & Streicher 1994:
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

Hofmann & Streicher groupoid model

$\vdash A : type$ - groupoid A
$\vdash x : A)$ - object x of groupoid A
$Id_A(x, y) : type$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
(no reason to be trivial unless x = y!)

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

HoTT: the idea

Types in MLTT can be modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in category theory.

Homotopy model of MLTT

- In the groupoid model of MLTT groupoids are *fundamental groupoids* (i.e., groupoids of paths) of topological spaces .
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

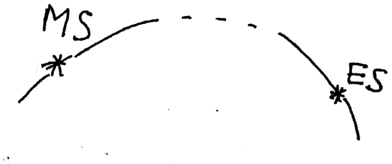
Propositional reduction as truncation

Each type is "made into" a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

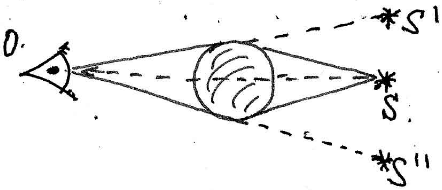
Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as constructive in the above sense (under certain conditions also in a stronger sense of being Turing-computable).

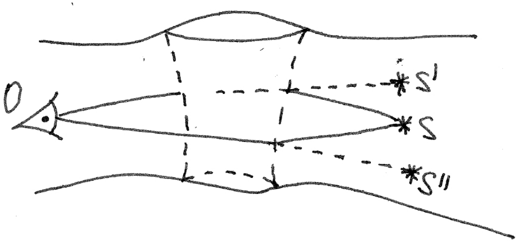
Identity through time



Gravitational lensing

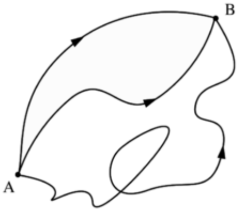


Wormhole lensing



Quantum trajectories

truncation: quantum $\longrightarrow$ classical



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- HoTT provides a modern formal framework for this traditional variety of axiomatic method;
- The traditional informal version of constructive axiomatic architecture already proved effective in physics (Newton, Clausius). The modern formal version of this architecture has better chances to be successfully used in science (in particular in computer-based Knowledge Representation systems) than the standard architecture.

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THANK YOU!