Streamlined Argument
Problem
Internal Logic
Homotopy Type theory
Constructive Axiomatic Method
Concluding Remarks

Internal Logic, Axiomatic Method and Constructive Knowledge Representation

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Problem:

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Problem:

- ▶ The received notion of axiomatic theory fails to adequately account for a large class of scientific theories based on the idea of (mathematical) *modeling* of Nature.

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Problem:

- ▶ The received notion of axiomatic theory fails to adequately account for a large class of scientific theories based on the idea of (mathematical) *modeling* of Nature.
- ▶ Such theories are particularly important and particularly widespread in applied sciences and in engineering.

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Problem:

- ▶ The received notion of axiomatic theory fails to adequately account for a large class of scientific theories based on the idea of (mathematical) *modeling* of Nature.

- ▶ Such theories are particularly important and particularly widespread in applied sciences and in engineering.

- ▶ Most (?) of the existing Knowledge Representation technologies are based on the idea of axiomatic presentation of knowledge and assume the received notion of (formal) axiomatic method in some form. As a result they inherit the same defect and fail to represent a model-based knowledge adequately and effectively.

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Problem:

- At the same time computational modeling techniques (including those based on mathematically-laden scientific theories) are highly developed and highly successful (e.g. Computational Flow Dynamics).

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Problem:

- At the same time computational modeling techniques (including those based on mathematically-laden scientific theories) are highly developed and highly successful (e.g. Computational Flow Dynamics).

- To make such a model-based (and already effectively computer-assisted) scientific and technological knowledge into a subject of digital KR technologies is a widely open problem.

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Problem:

▶ If the axiomatic approach works so badly why it deserves a discussion?

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Problem:

- ► If the axiomatic approach works so badly why it deserves a discussion?
- ► Because something of the sort is certainly needed since KR systems are supposed to support *reasoning* (automatic and/or human-assisted) in some form.

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Problem:

- ▶ If the axiomatic approach works so badly why it deserves a discussion?
- ▶ Because something of the sort is certainly needed since KR systems are supposed to support *reasoning* (automatic and/or human-assisted) in some form.
- ▶ If the *standard* axiomatic approach fails to perform the task properly (as in the aforementioned contexts) it needs some improvements or perhaps some replacement.

What a philosopher can do about it?

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What a philosopher can do about it?

- Revise the received view on scientific knowledge and knowledge in general, which lies behind the received version of axiomatic method;

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What a philosopher can do about it?

- ▶ Revise the received view on scientific knowledge and knowledge in general, which lies behind the received version of axiomatic method;

- ▶ In particular, rethink the place, the role and the scope of *logic* in epistemic matters;

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What a philosopher can do about it?

- ▶ Revise the received view on scientific knowledge and knowledge in general, which lies behind the received version of axiomatic method;

- ▶ In particular, rethink the place, the role and the scope of *logic* in epistemic matters;

- ▶ Develop an alternative view, which makes a better justice to the existing scientific practices (without giving up the normative function of epistemological ideas);

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What a philosopher can do about it?

- ▶ Revise the received view on scientific knowledge and knowledge in general, which lies behind the received version of axiomatic method;
- ▶ In particular, rethink the place, the role and the scope of *logic* in epistemic matters;
- ▶ Develop an alternative view, which makes a better justice to the existing scientific practices (without giving up the normative function of epistemological ideas);
- ▶ Motivate and guide new technical developments.

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What Causes the Problem?

In a standard (formal) axiomatic theory axioms provide a
propositional *description* of its models, i.e., they provide a necessary
and sufficient *condition* for being a model of this given theory.
However the axioms give no clue as to *how* the class (or at least
some) of such models can be found or built.

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What Causes the Problem?

The usual answer is that the models are found or built *first* - by
empirical trials and errors, intuitive guesses and whatnot - and the
corresponding axiomatic description of these models comes *later* as
a way of putting these pieces of knowledge (i.e., the informal
fragmentary models) into a logical order.

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What Causes the Problem?

This answer is NOT satisfactory and does NOT make the full justice to the successful scientific practice. It ignores the fact that successful scientific theories (e.g. Classical Mechanics or Flow Dynamics) allow for *generating* specific models from basic elementary pieces according to certain constructive rules.

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What Causes the Problem?

Even if in a standard axiomatic setting such a procedure can be
mimicked by a logical inference, in this later form the procedure is
usually computationally by far more costly or even unfeasible.

Such a logical mimicking, generally, causes a loss of relevant
information about the models.

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What Causes the Problem?

This is hardly surprising if one thinks about logical inferences (=
generating theorems) from axioms on equal footing with generating
specific models from basic elementary models.

From this point of view the standard idea of axiomatic presentation
amounts to finding a *universal* generative structure which serve for
all possible epistemic purposes.

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What Causes the Problem?

Playing with logical calculi used in axiomatic presentations of specific theories (tense logic, epistemic logic, quantum logic etc.) is a way to cope with the universality problem.

But neither of these alternative or more specific logics supports a generative structure for *models*.

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A Prospective Solution

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A Prospective Solution

▶ Use the idea of *internal* logic (as in Topos theory)

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A Prospective Solution

- ▶ Use the idea of *internal* logic (as in Topos theory)
- ▶ Use the concept of *constructive* axiomatic theory, which treats propositions on equal footing with non-propositional types (but contrary to the popular Carry-Howard paradigm makes a clear distinction between the two!)

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A Prospective Solution

- ▶ Use the idea of *internal* logic (as in Topos theory)
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- ▶ Use Homotopy Type Theory and the idea of Univalent Foundations as a motivating example (at least)!

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A Prospective Solution

- ▶ Use the idea of *internal* logic (as in Topos theory)
- ▶ Use the concept of *constructive* axiomatic theory, which treats propositions on equal footing with non-propositional types (but contrary to the popular Carry-Howard paradigm makes a clear distinction between the two!)
- ▶ Use Homotopy Type Theory and the idea of Univalent Foundations as a motivating example (at least)!
- ▶ Write a code! (cf. AGDA)

Tenets of Standard Axiomatic Method

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- ▶ Theories a sets of propositions. The core scientific knowledge is propositional or at least can be presented in a propositional form.

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Tenets of Standard Axiomatic Method

- ▶ Theories a sets of propositions. The core scientific knowledge is propositional or at least can be presented in a propositional form.

- ▶ Axiomatic theories are theories generated by (subsets of) their special propositions called *axioms* through *logical inferences*.

Tenets of Standard Axiomatic Method

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Tenets of Standard Axiomatic Method

- Logical inferences are truth-preserving: true axioms entail true theorems. Truth-values of axioms (as well as theorems) depend of their semantic interpretation.

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Tenets of Standard Axiomatic Method

▶ Logical inferences are truth-preserving: true axioms entail true theorems. Truth-values of axioms (as well as theorems) depend of their semantic interpretation.

▶ The language used for writing down axioms contains logical and non-logical terms. In a given theory the *logical* semantic is fixed in advance (as a part of logical prerequisites) while non-logical terms allow for multiple interpretations. This feature allows one and the same theory to have many different models.

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Note

The distinction between logical and non-logical terms is a *semantic* distinction between two different *types* of symbols. Nevertheless it is often introduced "as if" it would be a part of syntax. As a result the issue of *logical* semantics is sharply separated from the model theory of a given theory. There is no *technical* reason for separating these things, this separation comes from philosophy!

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Programs with Common Sense: McCarthy 1958

[I think of] programs to manipulate in a suitable formal language
(most likely a part of the predicate calculus) common instrumental
statements. The basic program will draw immediate conclusions
from a list of premises. These conclusions will be either declarative
or imperative sentences. When an imperative sentence is deduced
the program takes a corresponding action.

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Logicism vs. Anti-Logicism Debate

Anti-Logicist Objections (Bar-Hillel et al.): Logicism is too expensive and unfeasible (even if it is good in the ideal world).

Strategy: making computers more friendly to the natural language and common-sense reasoning (an impact of Analytic Philosophy?)

My view: the strategy is wrong. The real reason of failure of the standard axiomatic method is that it represents the *scientific* but not just the common-sense knowledge wrongly. Logicism is bad for science even if it is good for the common sense!

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Example: Computational Flow Dynamics
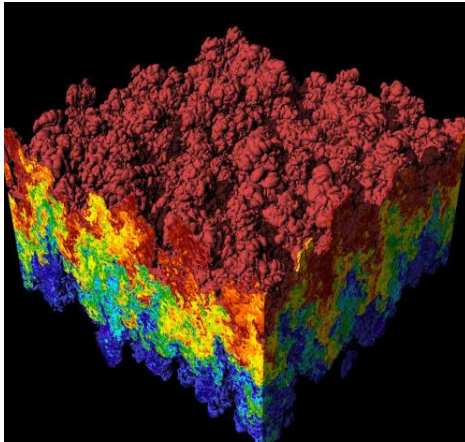
Theoretical Background: Navier-Stokes Equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial[\rho u_i u_j]}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i \tag{2}$$

$$\frac{\partial(\rho e)}{\partial t} + (\rho e + p)\frac{\partial u_i}{\partial x_i} = \frac{\partial(\tau_{ij}u_j)}{\partial x_i} + \rho f_i u_i + \frac{\partial(\dot{q}_i)}{\partial x_i} + r \tag{3}$$

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Computational Flow Dynamics

Computer Simulation: WATER RUNNING!

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How to make these or similar computations in a standard axiomatic setting in the form of logical inferences?

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some ZF axioms and modus ponens

Extensionality:
$$x = y \quad \Leftrightarrow \quad \forall z(z \in X \ \Leftrightarrow \ z \in y)$$

Pairing:
$$\exists u \forall z(z \in u \quad \Leftrightarrow \quad (z = x) \vee (z = y))$$

Union:
$$\forall u \exists v[\, x \in v \quad \Leftrightarrow \quad \exists w(w \in u \& x \in w)$$

$$\bullet \ \bullet \ \bullet \ \bullet \ \bullet$$

Modus Ponens:
$$P \rightarrow Q, P \vdash Q$$

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What is a formal counterpart of Navier-Stokes equations?

It is a finite but nevertheless merely *ideal* object X (like $10^{10^{10}}$) that can be studied mathematically (along with $10^{10^{10}}$). A mathematical study of X may shed some light on the Navier-Stokes and its theoretical environment.

HOWEVER X cannot help to solve or otherwise use these equations in anything like the usual sense of the word.

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How X is defined?

and how one can learn its properties?

One may tell a plausible story showing that a formal reconstruction of Navier-Stokes equations is possible "in principle". X is defined as the result of this informal (sic!) procedure. The procedure is not supposed to be effective except some artificial simple cases. In that respect X is quite a typical *mathematical* object (compare again with $10^{10^{10}}$)

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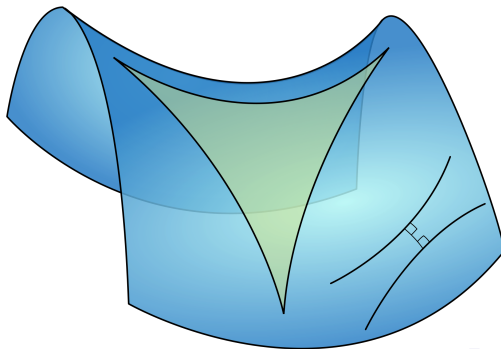Constructive Axiomatic Method
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Foundations

The claim according to which X presents the Navier-Stokes and its
theory in *the* standard well-founded form is a strong *philosophical*
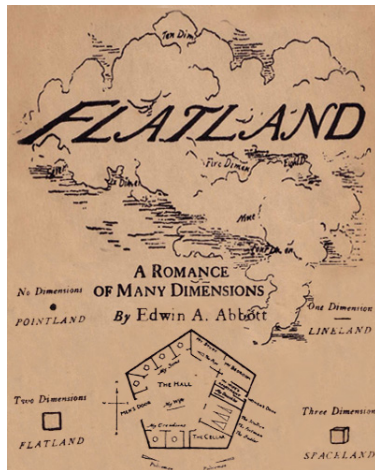claim, which is open to philosophical objections and doubts.

Without going into a thorough discussion on foundations of
mathematics one may only claim that X is a very special
representation of Navier-Stokes, which proves useful for logical
analysis (of a very particular sort) but certainly NOT for a
all-purpose representation of this theory and the associated
mathematical knowledge.

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Internal geometry: Gauss, Riemann, Einstein

Geometrical space is not an universal vehicle of all geometrical objects. Every object is a space on its own, which maps (and, in some cases embeds) into many other spaces.

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Abbott 1884

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Lawvere 1970: geometrical logic

[A] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions. [..] [I]n a sense logic is a special case of geometry.

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Elementary Theory of Category of Sets (ETCS)

Lawvere 1964

<u>Idea</u> (back to von Neumann late 1920-ies): functions instead of $\in$'s

<u>Remark:</u> everything stems from Lawvere's Thesis of 1963

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ETCS 1: ETAC

Elementary Theory of Abstract Categories

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i,j = 0,1$
- E2) $(\Gamma(x,y;u) \wedge \Gamma(x,y;u')) \Rightarrow u = u'$
- E3) $\exists u \Gamma(x,y;u) \Leftrightarrow \Delta_1(x) = \Delta_0(y)$
- E4) $\Gamma(x,y;u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x),x;x) \wedge \Gamma(x,\Delta_1(x);x)$
- E6) $(\Gamma(x,y;u) \wedge \Gamma(y,z;w) \wedge \Gamma(x,w;f) \wedge \Gamma(u,z;g)) \Rightarrow f = g$

E1)-E4): bookkeeping (syntax); 5): identity; 6): associativity

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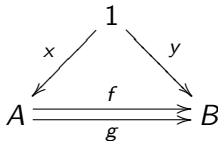
ETCS 2: topos (anachronistically):

- ▶ finite limits;
- ▶ Cartesian closed (CCC): terminal object (1), binary products, exponentials;
- ▶ subobject classifier

$$
\begin{array}{ccc}
U & \xrightarrow{\;!\;} & 1 \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle true} \\
X & \xrightarrow[\chi U]{} & 2
\end{array}
$$

for all p there exists a unique χU that makes the square into a pullback

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ETCS 3: well-pointedness

for all $f, g : A \to B$, if for all $x : 1 \to A$ $xf = xg = y$ then $f = g$

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ETCS 4: NNO

Natural Numbers Object: for all t', f there exists unique u

$$
\begin{array}{ccc}
1 \xrightarrow{\ t\ } & \mathbb{N} \xrightarrow{\ s\ } & \mathbb{N} \\
\ \ \downarrow{\scriptstyle u} & & \ \ \downarrow{\scriptstyle u} \\
{\scriptstyle t'}\searrow & & \\
A \xrightarrow{\ f\ } & A &
\end{array}
$$

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ETCS 4: Axiom of Choice

Every epimorphism splits:
If $f : A \to B$ is epi then there exists mono $g : B \to A$ (called
section) such that $g \odot f = 1_B$

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The idea of internal logic: Prehistory

- ▶ Boole 1847, Venn 1882: propositional logic as algebra and mereology of (sub) classes (of a given universe of discurse); logical diagrams

- ▶ Tarski 1938 topological interpretation of Classical and Intuitionistic propositional logic

Remark: Consider the similarity with Hegel's approach: logic is <u>not</u> god-given but appears as a feature of the given subject-matter. However in Boole, Venn and Tarski such an internal treatment concerns <u>only</u> propositional logicЙ

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The idea of internal logic: CCC

- ▶ Lawvere 1969: CCC is a common structure shared by (1) the simply typed λ-calculus (Schönfinkel, Curry, Church) and (2) Hilbert-style (and Natural Deduction style) Deductive Systems (aka Proof Systems).

- ▶ The fact that such a common structure exists is often *misleadingly* called *Curry-Howard correspondence* or *Curry-Howard isomorphism* (my upcoming talk of 14th March)

- ▶ The CCC structure is *internal* for *Set* BUT is more general: *Cat* (of all *small* categories) is another example; any *topos* is CCC.

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Internalization of Existential Quantifier

The *left* adjoint to the substitution functor f^* is functor

$$\exists_f : \ B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of X) into
$P \in B(Y)$ (subset of Y) consisting of elements $y \in Y$, such that
there exists some $x \in R$ such that $y = f(x)$; in (some more)
symbols

$$\exists_f(R) = \{y | \exists x(y = f(x) \wedge x \in R)\}$$

In other words $\exists_f$ sends R into its *image* P under f. One can
describe $\exists_f$ by saying that it transformes $R(x)$ into
$P(y) = \exists_f x P'(x, y)$ and interpret $\exists_f$ as the usual existential
quantifier.

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nternalization of Universal Quantifier

The *right* adjoint to the substitution functor f^* is functor

$$\forall_f : \ B(X) \longrightarrow B(Y)$$

which sends every subset R of X into subset P of Y defined as follows:

$$\forall_f(R) = \{y | \forall x(y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$.

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Moral

A higher-order logic is <u>also</u> internal for *Set* and ... toposes!

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CS rule of thumb

Curry-Howard Correspondence: Propositions-as-Types

MLTT: two identities

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MLTT: two identities

- Definitional identity of terms (of the same type) and of types:
  $x = y : A$; $A = B : type$ (substitutivity)

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MLTT: two identities

- Definitional identity of terms (of the same type) and of types:
  $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

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MLTT: extensional versus intensional

- Extensionality: Propositional identity implies definitional identity (ex. LCCC)

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MLTT: extensional versus intensional

- ▶ Extensionality: Propositional identity implies definitional identity (ex. LCCC)
- ▶ First intensional (albeit 1-extensional) model: Hofmann & Streicher 1994:
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

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Hofmann & Streicher groupoid model

$\vdash A : type$ - groupoid A

$\vdash x : A$) - object x of groupoid A

$Id_A(x, y) : type$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
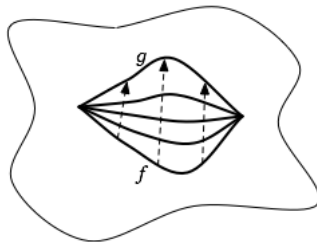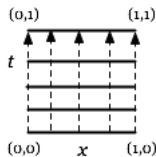
(no reason to be trivial unless x = y!)

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MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

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HoTT: the idea

Types in MLTT can be modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in category theory. (Voevodsky circa 2008).

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Path Homotopy

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Homotopy model of MLTT

- ▶ In the groupoid model of MLTT groupoids are *fundamental groupoids* (i.e., groupoids of paths) of topological spaces .
- ▶ Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

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Propositions-as-**Some**-Types !

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Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

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Propositional reduction as truncation

Each type is "made into" a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

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- ▶ Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.

- ▶ These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.

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Constructive axiomatic theory

contains rules for generating non-propositional objects from simple
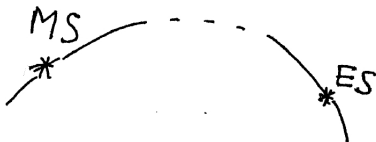elements along with rules for generating theorems from axioms

Examples: HoTT, Euclid's geometry (mind Problems and
Postulates!), Newton's Principia

Streamlined Argument
Problem
Internal Logic
Homotopy Type theory
Constructive Axiomatic Method
Concluding Remarks

Internal constructive axiomatization

building a mathematical model (of a physical phenomenon,
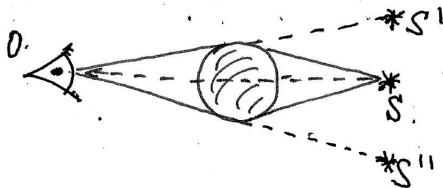technical product, etc.) along with its internal language

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Toy Models with HoTT

Identity through time:

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Toy Models with HoTT

Gravitational lensing

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Toy Models with HoTT

Wormhole lensing

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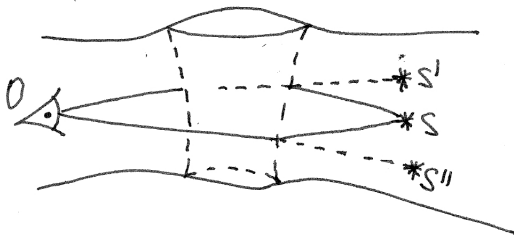
Toy Models with HoTT

Quantum trajectories (truncation: quantum $\rightarrow$ classical)

Concluding Remarks:

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▶ The idea of constructive axiomatic method is deeply rooted in history and proved effective for representing a model-based knowledge;

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▶ The idea of constructive axiomatic method is deeply rooted in history and proved effective for representing a model-based knowledge;

▶ HoTT provides a modern mathematical framework for this traditional variety of axiomatic method; however the same approach can be possibly realized also in some other mathematical settings.

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▶ HoTT provides a modern mathematical framework for this traditional variety of axiomatic method; however the same approach can be possibly realized also in some other mathematical settings.

▶ The constructive axiomatic architecture has better chances to be useful in the modern computer-based Knowledge Representation than the standard Hilbert-style axiomatic architecture.

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THE END