Constructive Knowledge
(logical and epistemological aspects)

Proceedings of seminar in the IP RAS, Moscow

organized with a support of the Russian Foundation for Basic Research grant 16-03-00364

Collected and edited by Andrei Rodin

Institute of Philosophy of Russian Academy of Sciences 2018

Contents

1 Scope and Aim

We call knowledge constructive if it includes an explicit specification of such associated epistemic procedures as capturing, verification, presentation, transmission, revision, and application of the given knowledge. Such a concept of constructive knowledge does not assume the social or cognitive constructivism according to which all knowledge is a social (resp. cognitive) construal independent of any real relation to its object. We study formal logical and epistemological aspects of various epistemic procedures related to knowledge belonging to a wide spectrum of fundamental and applied disciplines. A special focus is made on recent and prospective technologies of knowledge representation and knowledge management.

2 Serge P. Kovalyov (Institute of Control Problems RAS) Machine Intelligence in Engineering of Axiomatic Systems

Rigorous solutions to applied problems are relevant in many areas of technology. Notorious examples include verification of embedded software, axiomatic design of complex manufacturing products, virtual simulation of electronic equipment functioning, etc. [1]. However, in most cases, employing such approaches in real-world projects involves costs that significantly excess the visible useful results. One of the major reasons for this is the poor predictability of activities related to the application of the axiomatic method. It is unclear how to build practically useful axiomatic descriptions of domains without exceeding specified deadlines and budgets. Employing powerful computer tools is considered as a natural approach to solve this problem. Data mining tools are developed to accumulate large amounts of domain data in machine-readable form and process them on computers in order to extract the patterns of interrelations of a general nature that permit extrapolation beyond known data. With the help of domain experts, axioms and inference rules can be selected among patterns to form a deductive domain model as an axiomatic system. To apply axiomatic systems in practical problem solving, other computer tools are developed, viz. provers that are able to verify validity of any statements about the domain by automatic inference. As a result, the promising highly automated mode of application of the axiomatic method emerges, which deserves to be identified as a special branch of knowledge engineering, called engineering of axiomatic systems.

A variety of methods and tools of machine intelligence are used in engineering of axiomatic systems. Known illustrative examples include:

- recovering axioms in the course of automated logical inference on the so-called J-calculus [2];
- Inductive Logic Programming [3];
- domain ontology engineering [4];
- identifying the logical basis of the domain deduction by means of machine learning [5];
- distilling features into concepts (Meta-Interpretive Learning) [6].

The last two of these examples are based on the technology of deep neural networks that currently undergoes very intense development. There is even an opinion that the classical means of knowledge representation, based on explicit symbolic expression of the facts and laws, are hopelessly outdated and will soon be replaced by neural networks that manage implicit knowledge in a distributed form. However, distributed knowledge is unreliable, hard to verify, and prone to misrepresentation. Experiments are known when perfectly recognized images were practically imperceptibly perturbed in a special way, which was calculated in accordance with the "white spots" of the training sample, causing arbitrarily changes in the classifier's output [7]. So neural networks can greatly enhance, but not replace the axiomatic knowledge management tools.

All of these approaches, both formal and neural network based, are intended for use within a single domain. However, the current level of technological development is characterized by a multidisciplinary nature of the manufactured products: the products are complex systems that consist of components taken from several different domains. In order to build a coherent formal description of such a product, mechanisms are needed to integrate domain-specific axiomatic systems into sound holistic bodies of knowledge. Some time ago the ontology engineering "naively" recommended to rely on the public nature of knowledge that allegedly admit direct unification as soon as powerful referential capabilities are provided. However, such recommendation fails in practice due to the presence of antagonistic (ontological in the philosophical sense) contradictions between the diverse participants of the product life cycle.

Fortunately, holistic heterogeneous product descriptions useful for practitioners in systems engineering are typically restricted to certain specific viewpoint, or aspect, that can be identified in each component. The body of axiomatic knowledge about the aspect is sound; hence inter-component contradictions are easily detected, and resolved or left beyond the description. For example, the international standard IEC 81346-1:2009 "Industrial Systems, Installations and Equipment and Industrial Products - Structuring Principles and Reference Designations" specifies such aspects as function, material embodiment, location.

The processes of synthesizing the holistic product descriptions in an aspect are convenient to formally describe using the category-theoretic representation of axiomatic systems which was developed by H. Graves and others [8]. Let C be a category that represent the axiomatic system of the aspect, let I be the scheme (the shape) of the diagram that represents the structure of the complex product, and let $D_i, i \in I$ be the family of categories that represent axiomatic systems of product components. For each component i, the functor $F_i : D_i \to C$ is given that determines the rule to extract the target aspect from descriptions of the component. A particular holistic description of the product in the aspect C is obtained by choice of a family of objects $A_i \in D_i, i \in I$, and a diagram $\Delta : I \to C$ that satisfies the condition $\Delta(i) = F_i(A_i)$ for each $i \in I$.

Procedures of systems engineering are formally described by transformations of such descriptions. To specify and explore such transformations, we construct the category with descriptions as objects. For morphisms of such a category, we naturally employ natural transformations of descriptions' diagrams induced by morphisms of the components. Specifically, a morphism of a description $((A_i, i \in I), \Delta)$ to a description $((A'_i, i \in I), \Delta')$ is any family of morphisms $f_i : A_i \to A'_i, i \in I$ (where each morphism f_i belongs to the category D_i) such that for every two points of the scheme $i, j \in I$ and every arrow $s : i \to j$ the following naturality condition holds:

$$F_j(f_j) \circ \Delta(s) = \Delta'(s) \circ F_i(f_i)$$

.

It is easy to verify that such choice of morphisms indeed leads to a category. We will denote it as $\downarrow\downarrow_I F$. It is noteworthy that this category can be obtained using universal constructions in the "category of all categories" $\mathbb{CAT}$:

Theorem 1 *The category $\downarrow\downarrow_I F$ is isomorphic to a vertex of the following pullback in $\mathbb{CAT}$.*

One particular case of this construction is well known in category theory. It occurs when the schema $0 \to 1$, that consists of two points 0, 1, and one nontrivial arrow from 0 to 1, is employed as I. In this case the family F is reduced to a pair of functors

$$F_0 : D_0 \to C \leftarrow D_1 : F_1.$$

The generated category $\downarrow\downarrow_I F$ is known under the name "comma category" [9] and denoted as $F_0 \downarrow F_1$. Having this in mind, we call an arbitrary category of the kind $\downarrow\downarrow_I F$ a *multicomma* category. Theorem 1 allows us to derive a number of properties of the multicomma category which are useful in formal analysis of systems engineering procedures. For instance, if the schema I is discrete (i.e. doesn't contain any nontrivial arrows), then the multicomma category $\downarrow\downarrow_I F$ is isomorphic to a product of categories $\Pi_{i \in I} D_i$ independently on the choice of functors F_i and the aspect C. In other words, all descriptions of a multicomponent product that does not impose any relationship between its components form a conventional Cartesian product of representations of these components, even without having to choose any common aspect. This matches the intuitively clear possibility to place any set of noninteracting things into a common "bag" that is only nominally called the holistic product.

Furthermore, if every functor F_i is an isomorphism (i.e. all components are fully specified in the aspect C), then the multicomma category $\downarrow\downarrow_I F$ is isomorphic to a category C^I that consists of all diagrams of the form I in the category C, i.e. any graph is a valid description of the product. In addition, it is possible to show that the multicomma category construction behaves naturally with regard to sums and products of the schemes of the structure. One can prove a number of other statements that characterize certain particular design decisions on the composition of complex products.

In general, nowadays advances in engineering of applied axiomatic systems fall significantly behind the expectations that arose in the past decades. We hope that the intense employment of methods of machine intelligence and higher algebra will reduce this gap, and ultimately turn the axiomatic method into a powerful tool to solve real-world problems of systems engineering.

References:

[1]. Kovalyov S.P., Rodin A.V. Axiomatic Method in Contemporary Science and Technology: Pragmatic Aspects // Epistemology Philosophy of Science. 2016. Vol. 47, No. 1., p. 153-169. [In Russian]

[2]. Vassilyev S.N., Zherlov A.K., Fedosov E.A., Fedunov B.E. Intellectual Control over Dynamic Systems. Fizmatlit, 2000. [In Russian]

[3]. Muggleton S., De Raedt L. Inductive Logic Programming: Theory and Methods // The Journal of Logic Programming. 1994. Vol. 19-20. P. 629-679.

[4]. Volker J., Haase P., Hitzler P. Learning Expressive Ontologies. In: Ontology Learning and Population: Bridging the Gap between Text and Knowledge. Frontiers in Artificial Intelligence and Applications. 2008. Vol. 167. P. 45-69.

[5]. Sakama C., Inoue K. Can Machines Learn Logics? // 8th International Conference on Artificial General Intelligence (AGI-15). Lecture Notes in Artificial Intelligence. 2015. Vol. 9205. P. 341-351.

[6]. Dai W.-Z., Muggleton S., Zhou Z.-H. Logical Vision: Meta-Interpretive Learning for Simple Geometrical Concepts // 25th International Conference on Inductive Logic Programming (ILP-2015). Kyoto, 2015.

[7]. Szegedy C., Zaremba W., Sutskever I., Bruna J., Erhan D., Goodfellow I.J., Fergus R. Intriguing Properties of Neural Networks. CoRR abs/1312.6199. 2013.

[8]. Graves H. Axiomatic Category Theory for Knowledge Representation // NIST Workshop on Computational Category Theory. NIST, 2015.

[9]. Mac Lane S. Categories for the Working Mathematician. Springer, 1978.

3 Andrei Rodin (Institute of Philosophy RAS) Models of Homotopy Type Theory and the Semantic View of Theories

1. Categorical Model Theory

Today's Categorical Model theory (CMT) stems from the functorial semantics of algebraic theories proposed by Lawvere in his thesis back in 1963 [Lawvere:1963]. This theory uses a family of concepts of model none of which can be called today fairly standard. This fact is evidenced by the continuing discussion in the Homotopy Type theory (HoTT) [UFP:2013] where presently there is no full agreement among the researchers in the field as to what counts as a model of this theory and what does not.

One approch relies on the concept of *classifying category* T freely generated from the syntax of the given theory. Then a model M is a functor $T \to C$ into the category of sets ($C = Set$) or another appropriate category. This functorial setting has an important universal property: up to the categorical equivalence T can be identified with the initial object in the functor category of T-models. This property allows one to think of a theory in this setting as being a "generic model" (Lawvere). Using this approach Awodey [Awodey:2015] defines for HoTT the concept of *natural* model .

Voevodsky [Voevodsky:2015] pursues a different approach, which involves the concept of *contextual category* (more recently - in a modified form of C-system) earlier proposed by Cartmell [Cartmell:1986]. The idea behind the concept of contextual category is that of a category, which fully encodes all relevant algebraic features of the given syntax. According to this approach those and only those categories, which fall under the corresponding definition of contextual category, qualify as models of given theory T. In this case the initiality property of the syntactic category $S(T)$ is not implied by any general theorem. The initiality conjecture for HoTT still stands open.

Finally, there is yet another approac in CMT, which involves the concept of *internal language* (aka *internal logic*) of a given category. It has been recently proposed to think of internal languages and syntactic categories in terms of adjoint functors between a category of theories and a category of categories as shown on the diagram below:

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

In this setting a model of given theory T in a certain ground category C is a functor (a morphism in the category of theories) of the form

$$M : T \to Lang(C)$$

which expresses the idea of representation of a given theory in the language of some other theory (such as a representation of some geometrical theory in the language of arithmetic).

These and other technical advances of CMT so far have no generally accepted epistemological underpinning, which might help one to orientate among multiple developments. It remains, generally, unclear whether or not the classical Tarskian notion of model based on the T-schema and its standard epistemological understanding can be helpful in CMT. In what follows I show that the classical Tarskian concept of model is not adequate for accounting for the model theory of HoTT in its existing form and propose a remedy. Then I argue that the proposed non-standard understanding of concepts of theory and model can be used for supporting a new version of the semantic of view of theories, which may help to bridge the persisting gap between the notion of model as it is used in logic, on the one hand, and the colloquial notion of model used elsewhere in science, on the other hand.

2. Modeling HoTT

I shall consider HoTT *without* the univalence axiom. In this case the syntax of HoTT is that of (the intensional version of) Martin-Lof's Constructive Type theory (MLTT). HoTT also involves a semi-formal interpretation of its syntax in the Homotopy theory: types are interpreted as spaces (more precisely, infinite-dimensional fundamental groupoids of such spaces) and terms are interpreted as points of these spaces. This interpretation helped to reveal a feature of MLTT's syntax, which earlier remained hidden. Namely, it has been observed that types in MLTT are stratified into the so-called *homotopy levels*. It is important to stress that this stratification is a robust mathematical fact but not just a matter of one's favorite informal interpretation of the given calculus. This stratification necessitates a revision of the informal "propositions-as-types paradigm" , which is popular in the Computer Science. It shows that only types of certain homotopic level (namely, of level (- 1) as defined in [UFP:2013] can be identified with propositions while the higher types should be interpreted differently. This revision implies, in particular, that HoTT cannot be coherently interpreted as a system of propositions or sentences; correspondingly, the Tarskian notion of model based on the T-schema and the satisfaction relation applies only to propositional types (and the corresponding rules) of HoTT but not to this theory as a whole.

MLTT is a system of formal rules without axioms. In the case of propositional types these rules can be called *logical* rules in the usual sense. When these rules are applied to the higher types they should be thought of as rules for constructing non-propositional objects. A *model* of MLTT-HOTT is an implementation of this system of rules in some background, where higher-order constructions play the role of truth-makers for their associated propositions. (A proposition associated with a given higher type T is obtained from T via its (-1)*truncation*). This basic interpretation agrees with all existing models of HoTT disregarding the subtleties mentioned above. An interesting epistemological question is this. Does the epistemic role of higher-order constructions in HoTT reduce to their role as truth-makers or there is something more to it? Since the truncation of higher types to propositional types, generally, involves a significant loss of structure, HoTT rather supports the second answer (unless one assumes that a major part of this theory is epistemically insignificant). In the following concluding section I provide an independent argument, which supports the same conclusion and explains the epistemic value of higher non-propositional structures in HoTT.

3. Semantic View of Theories: a Constructive Perspective

P. Suppes [Suppes: 2002] argued that a typical scientific theory should be identified not with any particular class of statements (formal or contentual) but rather with a certain class of models . On this basis Suppes and his followers designed a Bourbaki-style format of formal presentation where a scientific theory is presented through an appropriate class of itsset-theoretic models.Albeit such a Bourbaki-style presentation can be useful for purposes of logical and structural analysis, it appears to be useless as a practical tool, which may help working scientists to formulate and develop their theories in a formal setting [Halvorson:2015].

Such a limitation is hardly surprising given that the standard set-theoretic semantics of theories provides no formal means for building and operating with models other than by referring to the fact that a model in question satisfies such-and-such propositions. Differences in epistemological views on the roles of syntax and semantics affect the style of formal presentation but not its architecture. This is why in practice the usual *non-statement* aka *semantic* approach to the formalization of scientific theories demonstrates the same limitations as its syntactically oriented rival.

HoTT and its model theory provides a novel notion of theory, which does not reduce to a class of propositions but has a further higher-order non-propositional structure. The axiomatic basis of such a theory consists of a system of rules, which apply both at the propositional and non-propositional levels. I believe that such a broader concept of theory and its model better fits the colloquial counterparts of these notions in the scientific practice than the standard Tarskian notions. The main reason isthat a typical scientific theory involves a lot of *procedural* content, which is used in modeling; such procedures may comprise but typically do not reduce to the procedures of logical inference (if by the logical inference one understands here a procedure which inputs and outputs sentences).

Thus HoTT and CMT provide the semantic view of theories with new formal techniques; the renewed semantic view, in its turn, provides an epistemological background for possible applications of these techniques in science and Knowledge Representation.

References:

[1]. S. Awodey, Natural models of homotopy type theory, arXiv:1406.3219, 2015

[2]. J. Cartmell, Generalised Algebraic Theories and Contextual Categories, Annals of Pure and Applied Logic, 32, p. 209-243, 1986

[3]. H. Halvorson, Scientific Theories, forthcoming in the Oxford Handbook of Philosophy of Science

[4]. F.W. Lawvere, Functorial Semantics of Algebraic Theories. Ph.D., Columbia University, 1963

[5]. P. Suppes, Representation and invariance of scientific structures, Stanford, Calif. : CSLI Publication 2002

[6]. V. Voevodsky, Martin-Lof Identity Types in the C-systems Defined by a Universe Category, arXiv:1505.06446, 2015

[7]. Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics, Institute for Advanced Study (Princeton) 2013; available at http://homotopytypetheory.org/book/,

4 Danya Rogozin (Moscow State University) Curry-Howard Correspondence and Kolmogorov Complexity

1. Preliminaries

1. Decompressor (description method) is a partial function from binary strings to binary strings, $D : \Xi \to \Xi$. If $D(x) = y$ for some $x, y \in \Xi$, than we shall talk that ¡¡x is a description y by D¿¿.

2. By Kolmogorov complexity of binary string by some D we shall mean is the shortest lenght of x, which $D(x) = y$:

$$KS_D(y) = min\{l(x) \mid D(x) = y\} \tag{1}$$

Informally, Kolmogorov complexity is a minimal length of a program that generates current string. If our description method is optimal (the best decompressor), then we talk about Kolmogorov complexity of string itself.

2. Kolmogorov Complexity and Intuitionistic Logic

Let A is a set of binary strings (cardinal number of this set is not important). A will be a task, $x \in A$ will be a solution of this task. Following by the Shen-Vereshchagin method[1] we can define logical operations on this sets of string $(\wedge, \vee, \rightarrow)$:

$$A \wedge B = \{(a, b) \mid a \in A, b \in B\} \tag{2}$$

$$A \vee B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\} \tag{3}$$

$$A \rightarrow B = \{p \mid \forall x \in A\, [p](x) \in B\} \tag{4}$$

Given definitions are based on the next one interpretation of logical connectives. This method goes back to Kolmogorov:

$A \wedge B$ — we can prove A and B.

$A \vee B$ — prove either A or B

$A \supset B$ — proof of B reduces to proof of A.

In the first case, conjunction of tasks has defined as cartesian product of their own desicions. We are going to define Kolmogorov complexity of conjunction as Kolmogorov complexity of pair in general case $(KS(x, y) \leq KS(x) + 2logKS(x) + KS(y) + O(1), x \in A, y \in B)$.

In the second case, disjunction of tasks has defined as union of their desicions. Let $KS(x, y) := min(KS(x), KS(y)) + O(1)$.

In the third case, implication of tasks we will define with a conditional complexity: $KS(x \rightarrow y) := KS(y|x) + O(1)$, $KS(y|x)$ is a complexity of transforming of x to y.

Now we consider some cases:

1. $A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A)$. $KS((x \rightarrow y) \wedge (x \rightarrow y)) = KS(x \equiv y) = max(KS(y|x), KS(x|y)) + O(logKS(x, y))$.

2. $KS((x \wedge y) \rightarrow z) = KS(z|x, y) + O(log(x, y))$. By other hand, $KS(x \rightarrow (y \rightarrow z)) = KS(z|x, y) + O(log(x, y))$.

3. $KS((x \rightarrow z) \wedge (y \rightarrow z)) = max(KS(z|x), KS(z|y)) + O(logKS(x, y, z))$.

[1] *Vereshchagin N. Alexander Shen.* Logical operations and Kolmogorov Complexity. Theoretical Computer Science. Vol. 271(1), 2002. Pp. 125-129

3. Binary λ-Calculus

Lambda-calculus is a formal system (computational model) invited by A.Church in the beginning of 1930-s. Lambda-calculus expresses computational processes by using notions of application and abstraction. Let us define this system formally.

1. Variable x is a term;

2. If M and N are terms then (MN) is term (application rule);

3. If x is a variable and M is a term, then $\lambda x.M$.

At the next step we must introduce De Bruijn notation recursievly by following grammar:

$$e ::= n \mid \lambda.e \mid e\, e \tag{5}$$

Examples ($\Rightarrow$ means ¡¡this term has the next one notation in De Bruijn syntax¿¿):

$\lambda x.x \Rightarrow \lambda 0$

$\lambda x.\lambda y.x \Rightarrow \lambda\lambda 1$

$\lambda f.\lambda g.\lambda x.g(fx) \Rightarrow \lambda\lambda\lambda 1\ (2\ 0)$

$\lambda f.\lambda g.\lambda x.(fx)(gx) \Rightarrow \lambda\lambda\lambda(2\ 0)(1\ 0)$

$\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \Rightarrow \lambda(\lambda 1\ (0\ 0))(\lambda 1\ (0\ 0))$

Following by John Tromp[2] we consider the way of binary coding of De Bruijn notated lambda-terms.

1. $n := 1^{n+1}0$;

2. $\widehat{\lambda M} := 00\widehat{M}$;

3. $\widehat{MN} := 01\widehat{M}\widehat{N}$

Let us expand or previous example ($\Longrightarrow$ mean ¡¡this De Bruijn notated term has the next one binary string¿¿):

$\lambda x.x \Rightarrow \lambda 0 \Longrightarrow 0010$

[2] *Tromp J.*, Binary Lambda Calculus and Combinatory Logic, in Randomness And Complexity, from Leibniz To Chaitin, ed. Cristian S. Calude, World Scientific Publishing Company, October 2008.

$\lambda x.\lambda y.x \Rightarrow \lambda\lambda 1 \Longrightarrow 0000110$

$\lambda f.\lambda g.\lambda x.g(fx) \Rightarrow \lambda\lambda\lambda 1\,(2\,0) \Longrightarrow 0000000111001111010$

$\lambda f.\lambda g.\lambda x.(fx)(gx) \Rightarrow \lambda\lambda\lambda(2\,0)(1\,0) \Longrightarrow 00000001011110100111010$

$\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \Rightarrow \lambda(\lambda 1(00))(\lambda 1(00)) \Longrightarrow 0100000111011101100001110111 0110$

4. Simply Typed λ-Calculus

The most useful form of typed lambda calculus is a calculus with Curry-style typing form. The next one rules are defined for abstaction and application:

$$\frac{\Gamma, x:\phi \vdash M:\psi}{\Gamma \vdash (\lambda x.M):\phi \to \psi}(\to_{In}) \quad \frac{\Gamma \vdash M:\phi \to \psi \;\; \Delta \vdash x:\phi}{\Gamma, \Delta \vdash Mx:\psi}(\to_{El}) \tag{6}$$

Curry-Howard isomorphism[3] is a two-side correlation between computer programs (typed lambda-terms) and proofs in natural deduction style[4], which can be formulated by two principles: ¡¡proof-as-terms¿¿ and ¡¡propositions-as-types¿¿. The first principle claims that lambda-term codes natural-deduction proof, the second one claims that intuitionistically valid proposition corresponds to the inhabited type.

For example, we can prove $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$ in natural deduction.

1. $A \supset B$;
2. $B \supset C$;
3. A;
4. B — 1, 3, $\supset_{El}$;
5. C — 2, 4, $\supset_{El}$;
6. $A \supset C$ — 3, 5, $\supset_{In}$;
7. $(B \supset C) \supset (A \supset C)$ — 2, 6, $\supset_{In}$;

[3] *Sorensen M.H., Urzyczyn P.* Lectures on the Curry-Howard Isomorphism. — Amsterdam: Elsevier, 2006. Pp. 69 - 102.

[4] *Dummett M.* Elements of Intuitionism. The Second Edition. — Oxford: Oxford University Press, 2000. P. 88.

8. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$ — 1, 7, $\supset_{In}$.

But we can lambda-term of the next type $(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ by the same way:

1. $f : \alpha \to \beta$;
2. $g : \beta \to \gamma$;
3. $x : \alpha$;
4. $fx : \beta$;
5. $g(fx) : \gamma$;
6. $\lambda x.g(fx) : \alpha \to \gamma$;
7. $\lambda g.\lambda x.g(fx) : (\beta \to \gamma) \to \alpha \to \gamma$;
8. $\lambda f.\lambda g \lambda x.g(fx) : ((\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma)$

5. Further Research

The goal of a further research is a modifying of the Shen-Vereshchagin method. At the first, it's possible to use type-inference algorithm. Type-inference algorithm returns by given lambda-term its type for polynomial time (this algorithm is applied for type infering in functional languages, Haskell, OcaML, F♯, Idris, etc). That result belongs to Hindley, Milner and Damas. At the second, well-formed (from a type-theoretical point of view) lambda-term has type, which is in conformity with appropriate intuitionistically valid proposition. From a logical point of view, this fact means that we can get a proposition using the code of its own proof.

Without considering all the particulars and somewhat informally, we are going to try to define such a method. At the first step, we represent an arbitrary lambda-term to a binary string following Tromp's rule. Next thing we have to do, we have to get upper bound of Kolmogorov complexity for obtained string. After that, we apply Hindley-Milner algorithm to the given lambda-term. If term is well-formed, then obtained type is the single. Kolmogorov complexity of this type (or proposition, or task) is a Kolmogorov complexity of binary lambda-term. It means that we define complexity of formula (task) by its proof (decision). For example, we have some term $M : \phi \times \psi \to \tau$, then we can define complexity of M as $KS(M) = (z|x, y) + O(log(x, y, z)), z : \tau, x : \phi, y : \psi$ (according to the Shen-Vereshchagin definitions).

BHK-semantics suggests to consider the intuitionistic logic as the calculus of problems (or tasks). Curry-Howard correspondence gives us a formalization of the notion of solution of the task. The kernel of modifying of the Shen-Vereshchagin method is a using binary lambda-terms as solutions of a given problem.

The long-run objective is construction of the general method for defining of complexity of intuitionistic formulas using typed lambda-calculus and Curry-Howard correspondence.

References:

[1]. *Vereshchagin N. Alexander Shen.* Logical operations and Kolmogorov Complexity. Theoretical Computer Science. Vol. 271(1), 2002. Pp. 125-129

[2]. *Tromp J.*, Binary Lambda Calculus and Combinatory Logic, in Randomness And Complexity, from Leibniz To Chaitin, ed. Cristian S. Calude, World Scientific Publishing Company, October 2008.

[3]. *Sorensen M.H., Urzyczyn P.* Lectures on the Curry-Howard Isomorphism. Elsevier, 2006.

[4]. *Dummett M.* Elements of Intuitionism. The Second Edition. Oxford University Press, 2000.

5 Konstantin Shishov (Moscow State University) Reversible Logical Gates in Quantum Logic

1. Preliminaries

According to Moore's thesis, growth of computing power is limited by fundamental physical principles that underlie electronic computing circuits. One of these is the principle of Landauer, according to which the logic circuits, which are not *reversible* should produce heat in proportion to the number of erasure, in the process of computation. Accordingly, the computer scientists are looking for ways to overcome these limitations. One of these ways is introduction of the quantum principles into computational process, which called quantum computing.

However, quantum computing is developing quantum logic - non-classical logic, which involves the construction of logical systems describing quantum computing.

This paper is in line with the quantum logic, and aims to focus on *reversible* nature of logical operations. There exists a close connection between classical reversible computation and quantum computation, since all unitary quantum operations are necessarily reversible; therefore, reversible computing is a subset of quantum computing.

1. Quantum Bits

Consider the two-dimensional Hilbert $\mathbf{C}^2$, where any vector $|\varphi\rangle$ is represented by a pair of complex numbers. Let $B = \{|0\rangle, |1\rangle\}$ be the orthonormal basis for $\mathbf{C}^2$ such that

$$|0\rangle = (0,1); \quad |1\rangle = (1,0)$$

Definition 1. Qubit

A qubit is a unit vector $|\varphi\rangle$ of the space $\mathbf{C}^2$ such that

$$|\varphi\rangle = a_0|0\rangle + a_1|1\rangle,$$

where $a_0, a_1 \in \mathbf{C}^2$ and $|a_0|^2 + |a_1|^2 = 1$.

Futher we will use *x,y,z,...* as a variable ranging over the set {0,1}. At the same time $|x\rangle, |y\rangle, |z\rangle, ...$ will ranging over the basis $\{|0\rangle, |1\rangle\}$. The set of all vectors having the form $|x_1\rangle \otimes ... \otimes |x_n\rangle$, where "$\otimes$" represents the tensor product, represents a computational basis $\otimes^n \mathbf{C}^2$, such as:

$$\otimes^n \mathbf{C}^2 =_{Df} \mathbf{C}^2 \otimes ... \otimes \mathbf{C}^2$$

Now we define n-register.

Definition 2. n-register
A n-register is any unit vector $|\psi\rangle$ in the $\otimes^n \mathbf{C}^2$.

Obviously, the computational basis $\otimes^n \mathbf{C}^2$ can be represented by sequences of $|0\rangle$ and $|1\rangle$ length $n : |101...011\rangle$. Such sequence represents a natural number $j \in [0, 2n-1]$ in binary notation. We obtain that any unit vector of $\otimes^n \mathbf{C}^2$, can be shortly expressed as:

$$\sum_{j=0}^{2^n-1} a_j|j\rangle$$

In accordance with this we will call any vector that is either qubit or an n-register, a quregister.

2. Logical Gates as Reversible Gates

The model of reversible computation has to fulfil these two conditions: the number of inputs and outputs of the function f has to be the same, and f the model has to have a one-to-one Boolean function. Likewise, we can pose the problem of universality as before, and ask for a set of universal reversible logic gates that can simulate arbitrary reversible Boolean functions.

One of the most famous reversible logical gates is a *Toffoli gate* which is named after its creator and is also known as CCNOT.

Definition 3. Toffoli gate $T^{(1,1,1)}$

The Toffoli gate $T^{(1,1,1)}$ is the linear operator $T^{(1,1,1)} : \otimes^3\mathbf{C}^2 \rightarrow \otimes^3\mathbf{C}^2$ that is defined for any element $|x\rangle \otimes |y\rangle \otimes |z\rangle$ of the basis as follows:

$$T^{(1,1,1)}(|x\rangle \otimes |y\rangle \otimes |z\rangle) = |x\rangle \otimes |y\rangle \otimes |min(x,y) \oplus z\rangle,$$

where "$\oplus$" represents the sum modulo 2.

Obviously, logical gate $T^{(1,1,1)}$ can be interpreted as a simple truth-table that transforms triples of bits (qubits) to triples of bits (qubits).

$$\begin{array}{c}
|0,0,0\rangle \longrightarrow |0,0,0\rangle \\
|0,0,1\rangle \longrightarrow |0,0,1\rangle \\
|0,1,0\rangle \longrightarrow |0,1,0\rangle \\
|0,1,1\rangle \longrightarrow |0,1,1\rangle \\
|1,0,0\rangle \longrightarrow |1,0,0\rangle \\
|1,0,1\rangle \longrightarrow |1,0,1\rangle \\
|1,1,0\rangle \longrightarrow |1,1,1\rangle \\
|1,1,1\rangle \longrightarrow |1,1,0\rangle
\end{array}$$

It is proved that this is an universal logic gate. That is, this can be used to replace all other logical connectives. For example, conjunction:

$$\mathbb{AND}(|\varphi\rangle, |\psi\rangle) =_{Df} T^{(1,1,1)}(|\varphi\rangle \otimes |\psi\rangle \otimes |0\rangle)$$

...and it's truth-table:

$$\begin{array}{c}
(|0\rangle, |0\rangle) \longrightarrow T^{(1,1,1)}(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \\
(|0\rangle, |1\rangle) \longrightarrow T^{(1,1,1)}(|0\rangle \otimes |1\rangle \otimes |0\rangle) = |0\rangle \otimes |1\rangle \otimes |0\rangle \\
(|1\rangle, |0\rangle) \longrightarrow T^{(1,1,1)}(|1\rangle \otimes |0\rangle \otimes |0\rangle) = |1\rangle \otimes |0\rangle \otimes |0\rangle \\
(|1\rangle, |1\rangle) \longrightarrow T^{(1,1,1)}(|1\rangle \otimes |1\rangle \otimes |1\rangle) = |1\rangle \otimes |1\rangle \otimes |1\rangle
\end{array}$$

Thus defined, the conjunction will be just as reversible as the Toffoli gate itself is reversible.

All this happens in the simplest situation, when one is only dealing with elements of the basis (in other words, with precise pieces of information). Let us examine the case where the function AND is applied to arguments that are superpositions of the basis-elements in the space $\mathbf{C}^2$. Consider the following qubit pair:

$$|\varphi\rangle = a_0|0\rangle + a_1|1\rangle \text{ and } |\psi\rangle = b_0|0\rangle + b_1|1\rangle\text{,when:}$$

$$\mathbb{AND}(|\varphi\rangle, |\psi\rangle) = (a_1b_1|1,1,1\rangle + a_1b_0|1,0,0\rangle + a_0b_1|0,1,0\rangle + a_0b_0|0,0,0\rangle)$$

Like in a classical logic the function $\mathbb{AND}$ corresponds to the values (1,1,1), (1,0,0), (0,1,0), (0,0,0). But, unlike classical logic, each case is accompanied by a complex number that represents a quantum amplitude - a characteristic reflecting the degree of probability with which the triple can be realized. For example, recording $|a_1b_1|^2$ determines the likelihood that both qubits are unity, and therefore, their conjunction is equal to unity.

Now consider how quantum logic introduces negation. First, it is necessary to consider the function $\mathbb{NOT}$, which is a generalization of the classical negation that negates the value of the last element in the base vector. In this vector notation it looks as follows: if $|x_1, ..., x_n\rangle$ is a vector in the computational basis $\otimes^n\mathbf{C}^2$, the result of the application of $\mathbb{NOT}$ is $|x_1, ..., 1 - x_n\rangle$

In case of denial to one qubit, the function $\mathbb{NOT}$ becomes a single function that associates elements-arguments from the space $\mathbf{C}^2$ elements, the values of $\mathbf{C}^2$.

$$\mathbb{NOT}^{(1)} =_{Df} (a_1|0\rangle + a_0|1\rangle)$$

It may be noted that the thus defined function $\mathbb{NOT}$ is a generalization of the classical negation.

In general, the function $\mathbb{NOT}$ is defined as follows:

$$\mathbb{NOT}^{(n)} : \otimes^n\mathbf{C}^2 \longrightarrow \otimes^n\mathbf{C}^2 \text{ and for all } |\varphi\rangle = \sum_{j=0}^{2^n-1} a_j|j\rangle \in \otimes^n\mathbf{C}^2$$

$$\mathbb{NOT}(|\varphi\rangle) = \sum_{j=0}^{2^n-1} a_j|x_{j_1}, ..., x_{j_{n-1}}, 1 - x_n\rangle$$

3. Quantum Logical Gates as Reversible Gates

In quantum computing and specifically the quantum circuit model of computation, a quantum logic gate is a basic quantum circuit operating on a small number of qubits. They are the building blocks of quantum circuits, like classical logic gates are for conventional digital circuits. Unlike many classical logic gates, quantum logic gates are reversible. However, it is possible to perform classical computing using only reversible gates. For example, the reversible Toffoli gate can implement all Boolean functions. This gate has a direct quantum equivalent, showing that quantum circuits can perform all operations performed by classical circuits.

The logical gates we have considered so far are, in a sense, generalizations of the classical logical connectives. A quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, our gates behave like classical truth-functions. We will now investigate genuine quantum gates that may transform classical registers into quregisters that are in superpositions.

Definition 4. Quantum logical gate

Quantum logical gate is an unitary operator, assuming arguments in a $\otimes^n \mathbf{C}^2$ values in $\otimes^n \mathbf{C}^2$.

One of the most significant genuine quantum gates is the square root of the negation (NOT), which will be indicated by $\sqrt{\mathbb{NOT}}$. As suggested by the name, the characteristic property of the gate $\sqrt{\mathbb{NOT}}$ is the following: for any quregister $(|\psi\rangle)$

$$\sqrt{\mathbb{NOT}}(\sqrt{\mathbb{NOT}}(|\psi\rangle)) = \mathbb{NOT}(|\psi\rangle)$$

A more general definition of $\sqrt{\mathbb{NOT}}$ is as follows:

$$\sqrt{\mathbb{NOT}}^{(1)} : \mathbf{C}^2 \longrightarrow \mathbf{C}^2 \text{ and for all } |\psi\rangle = a_1|0\rangle + a_0|1\rangle$$
$$\sqrt{\mathbb{NOT}}^{(1)}(|\psi\rangle) = \tfrac{1}{2}[(1+i)a_0 + (1-i)a_1]|0\rangle + \tfrac{1}{2}[(1-i)a_0 + (1+i)a_1]|1\rangle,$$

where i is an imaginary unit.

So $\sqrt{\mathbb{NOT}}^{(1)}$ translate two basic-station of qubit $|0\rangle$ and $|1\rangle$ into the superposition of this states.

Consider another "purely" quantum gate:

Definition 5. Hadamard-gate (H-gate)

For all qubit $n \geqslant 1$ Hadamard-gate $\otimes^n \mathbf{C}^2$ is a linear operator $H^{(2^n)}$ such as $|x_1, ..., x_n\rangle$ computational basis $\mathcal{B}^{(2^n)}$:

$$H^{(2^n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes 1\sqrt{2}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle$$

Thus it turns out that for $n = 1$ the use of the Hadamard-gate will have the form $H^{(2^n)}(|x\rangle) = H(|x\rangle)$ own case, while, as in the $n > 1$, we can see its connection with other logical gate: $n > 1$ $H^{(2^n)}(|x_1, ..., x_n\rangle) = I^{n-1}(|x_1, ..., x_{n-1}\rangle) \otimes H(|x_n\rangle)$

Hadamard-gate mapping $|0\rangle$-basis into $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and $|1\rangle$-basis in $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$, that corresponds to a rotation about the axis π. A double application Hadamard-gate will correspond to *identity gate*, which maps its input to the output unchanged.

4. Further Research.

This investigation presents an overview of some aspects of the quantum and reversible computation. It serves as a better understanding of the specific characteristics of quantum logic. In addition, in the process of studying the material we singled out some unresolved issues that form the scientific foundation for the future research. One of such problems is to compare unitary operations in quantum logic with modal operators. Is it possible to convert the work of processes in quantum logic to work with modal operators? Or, more broadly, is it possible to reduce quantum logic to modal logic?

References:

[1]. Cattaneo G., Dalla Chiara M.L., Giuntini R., Leporini R. *An unsharp logic from quantum computation* // International Journal of Theoretical Physics, 2004, Volume 43, Issue 7-8, pp 1803-1817

[2]. Dalla Chiara M.L., Giuntini R., Greechie R. *Reasoning in Quantum Theory*, Springer, 2004

[3]. Dalla Chiara M.L., Giuntini R. *Quantum logic* // Handbook of Philosophical Logic, Vol. 6, Kluwer Academic Publishers

6 Sergei Titov (Institute of Philosophy RAS) Data Science and Scientific Explanation

1. Issue

For the last years amount of available data of any kind for research grew up significantly, especially in domain of social sciences [Sagiroglu & Sinanc: 2013]. One can easily buy any amount of public data collected by such companies like GNIP or participate in data grant programs. Today we overcame lack of data about surrounding world - probably every human action is somehow digitalized and saved, every nature event is carefully documented. Key problem now is to decide what to do with all this data - incompetent research may lead us to absurd results [Bennett et al.: 2010] at best case. It may seem that such problem statement is speculative, but in this work we will try to show how data may affect the study.

<u>2. Data and Phenomena</u>

Woodward and Bogen argue about relations between data, phenomena and theory [Bogen & Woodward: 1988]. Main idea of the work is that phenomena is something that exists but is unobservable while data plays a role of evidence of the existence of such phenomena. In other words, Bogen and Woodward indue phenomena with ontological status.

Point here is that such phenomena is strictly unobservable as example authors demonstrate that regardless of technology we use it is impossible to know true value of melting point of lead. In case of the melting point we always end the research with some database of temperatures and we need to make decision about final value. From the Woodward and Bogen point of view this decision in a way is irrelevant because phenomena is in laws of nature" domain and some true value is actually in the data.

In a critique on Woodward and Bogen paper, MacAllister [McAllister: 1997] pays closer attention on this decion making. He claims that this decision is scientist personal choice so phenomena doesn't actually have ontological status. In this case deducing phenomena from a data become psychological issue - way of scientist observe data become essential for phenomena detection.

Longo and Calude [Calude & Longo: 2015] present formal restrictions in phenomena detection. They build argument upon "spurious correlations" which is based on Ramsey theory. Here we will briefly describe their argument.

Ramsey theory suggest that for every predefined structure there is big enough set where it can be guaranteedly found. For example. A special case is given by the following theorem due to van der Waerden: For any positive integers k and c there is a positive integer γ such that every string, made out of c digits or colors, of length more than γ contains an arithmetic progression with k occurrences of the same digit or color, i.e. a monochromatic arithmetic progression of length k [Shkredov:2006].

In case of big enough database we can find there arbitrary long progression of correlations in order to consider them relevant? Here contradiction emerge - reliability of correlation may be caused only by size of database - we devoted to found any correlations in big enough databases.

Longo and Calude define spurious correlations as a “correlation is spurious if it appears in a 'random' generated database”. Despite strictness of this definition it allows us to estimate number of such correlation using Kolmogorov theory. In the paper authors bring out calculations for correlations spuriousness” for 2048 symbols long string and get in result that overwhelming majority of possible correlations could be considered as spurious.

An interesting fact is that Kolmogorov theory and Ramsey theory in case of this argument have a intriguing contradiction: if we will take a long enough and completely random string (in Kolmogorov sense) it would have some predefined pattern according to Van der Waerden theorem. Can we consider such string truly random?

While this paper stops just on this statement there is one more step to be done: what to do with spurious correlations? This particular paper is not focused on such question and poof above may allow us only to argue about of impossibility of replacement science with data analysis.

3. Embracing Descriptive Science

Another view on data-intensive science lies beyond the problem of phenomena detection. If we agree that all patterns in data are some phenomenas we would run into another epistemological problem.

In his recent paper Pietsch suggest to distinguish vertical and horizontal type of science. First one is classical view on science where researches have some structure of concepts - from the most universal laws to practical and especial cases. These concepts are consistent and can be deduced from each other. Good example of vertical science structure is physics but it is the exception rather than the rule - the most of the knowledge areas doesn't fit in such standards, so there is horizontal science.

Horizontal model describe disciplines like social sciences or economics in which vertical model has failed. While vertical model rely on reduction and theoretical strictness horizontal model based on efficient analysis of big amount of data. Pietsch outline four main properties of horizontal model:

- Predictions are made from the data directly without casual structure of phenomena

- There is no need in high-abrstrat concepts because probably every aspect of the phenomena is already in data
- Lack of explanatory power in models
- Rare use of idealisations and simplifications

Pietsch [Pietsch:2015] suggests that large enough database contain every state of phenomena which change probabilistic inference into unambiguous one. Such procedure we can consider as a case of eliminative induction and transition from descriptive statistics to causal inference. So complete horizontal model will provide casual predictions like vertical does but with serious lack of explanatory power.

Napoleani and Panza [Napoleani & Panza:2011] provide similar view on future of special sciences (in Woodward sense). Like Pietsch they argue that deductive model have failed in this type of disciplines and suggest to concentrate scientific attention specifically on data analysis. They propose term of "agnostic science" which imply the idea of "turn over" relation of mathematical methods and empirical research. In classical science mathematical apparatus was usually developed specifically for every phenomena, but in "agnostic" science Napoleatani and Panza argue that research would seek already developed mathematical model for data which describes phenomena:

"Mathematics becomes perhaps the only domain in which to develop structural understanding, since such pretense is lost in the study of phenomena. Ideas are then forced upon the phenomenon in problem solving, only temporary, and with little expectations that go further than the solution of the problem. Scientific methods may become weak, but the mathematical language in which they are phrased will be increasingly complex, as we attempt to mould our desires, coarsely, upon reality."

Good example for such idea is hurricane prediction by neural networks. Despite the fact that we have physical model for this, predictions made by neural network with base only on data are much more precise. But the price of such efficiency is explanatory power. Despite of examples in the paper this approach begs the questions of possibility of further progress of the phenomenon research after applying independent developed structure.

<u>4. Conclusion</u>

We have considered two interrelated problems. The first problem appears in phenomena discovery in data - in large enough databases we could find any pre-defined pattern because only of the size of database. It make unreliable some of recent scientific findings - for example, false-positive results of fMRI - analysis [Eklund & Nichols & Knutsson : 2016]. The second problem appears in case of complex data analysis. There is a wide variety of methods which rely on model generation of data like neural networks or boosting algorithms. Even if one obtains fascinating results one looses an explanation of the studied phenomena.

In case of the first problem further work would be concentrated around Longo and Caulde argument - primary task is to check some known datasets for spurious correlation probability. In addition there are some ways to strengthen the argue, for example with use of more powerful theorems, like Semeredy theorem [Shkredov:2009].

The second problem raises the question of explanation power loss which could be very perspective from the philosophy of science point of view. Development of the new methodologies which could help to conduct explanatory research in data intensive science is essential for the science.

References:

[1]. Bennett C.M. et. al. Neural Correlates of Interspecies Perspective Taking in the Post-Mortem Atlantic Salmon: An Argument For Proper Multiple Comparisons Correction // J. Serendipitous Unexpected Results. 2010. . 1. 1. . 15.

[2]. Bogen J., Woodward J.F. Saving the Phenomena // Philos. Rev. 1988. . 97. 3. . 303352.

[3]. Calude C., Longo G. The Deluge of Spurious Correlations in Big // Found. Sci. 2015. . 1.

[4]. Eklund A., Nichols T.E., Knutsson H. Cluster failure: Why fMRI inferences for spatial extent have inflated false-positive rates // Proc. Natl. Acad. Sci. 2016. . 16.

[5]. McAllister J.W. Phenomena and patterns in data sets // Erkenntnis. 1997. . 47. 2. . 217-228.

[6]. Napoletani D., Panza M. Agnostic Science . Towards a Philosophy of Data Analysis // Found. Sci. 2011. 16. . 1-20.

[7]. Pietsch W. The Causal Nature of Modeling with Big Data // Philos. Technol. 2015. . 1-37.

[8]. Sagiroglu S., Sinanc D. Big data: A review // 2013 International Conference on Collaboration Technologies and Systems (CTS). , 2013. . 42-47.

[9]. Shkredov I.D. Semeredi Theorem and Problems of Arithmetical Progressions // Uspekhi Matematicheskikh Nauk 2006 v. 6, N 372, p. 111-179 [in Russian].

7 Vladimir L. Vasyukov Scientific Pluralism: Logics, Ontology, Mathematics

1.Introduction

Presently many philosophers and scientists are inclined to take a pluralistic position regarding scientific theories or methods. It is a common wisdom that the totality of natural phenomena cannot be possibly explained with a single theory or a single approach. (cf. [14]). Current debates on the scientific pluralism usually involve the 'Unity of Science' thesis first advanced by Neo-Positivists in the 1930-ies. According to this thesis "Laws and concepts of particular sciences have to belong to the one system and be reciprocally related. They have to form certain unified science with a common system of concepts (common language), separate sciences are just the members of it and their languages are parts of the common language" [15, p. 147-148].

In 1978 Patrick Suppes [28] in his presidential address to the Philosophy of Science Association claimed that the time for defending science against metaphysics (which he took to be the original rationale for the unity of science movement) had passed. Suppes argued that neither the languages of scientific disciplines nor their subject matters were reducible to one language and one subject matter. Nor was there any unity of method beyond the trivially obvious such as use of elementary mathematics.

The majority of philosophers of science were not particularly enthusiastic about Suppes's ideas. A noticeable exception was Nancy Cartwright and her collaborators who stressed the irreducible variety of scientific disciplines involved in solving concrete scientific problems. Later Cartwright [7] elaborated a pluralistic account of a 'dappled world' composed of a number of separate areas. Each particular area of this world is ruled by its own laws, so that this system laws form a loose patchwork, which does not reduce to a single compact system of fundamental laws. A similar view has been put forward by John Dupré [11] who also supports a pluralist metaphysical position called the "promiscuous realism".

One has to distinguish between the pluralism *in* science and the pluralism *about* science. At any stage of their development sciences typically use a variety of different approaches corresponding to different aspects of studied phenomena. They use various representational or classificatory schemes, various explanatory strategies, various models and theories, etc. This is a pluralism *in* science. The pluralism *about* science is a view according to which such a plurality of approaches in science is ineliminable as a matter of principle, and that it does not constitute any deficiency in knowledge. According to this view an analysis of meta-scientific concepts (such as theory, explanation, evidence) should take into consideration the possibility that in the long run the explanatory and investigative aims of science can be best achieved with a pluralistic science.

Modern scientific monism can be described as follows [14, p. x]:

- the ultimate aim of a science is to establish a single, complete, and comprehensive account of the natural world (or the part of the world investigated by the science) based on a single set of fundamental principles;
- the nature of the world is such that it can, at least in principle, be completely described or explained by such an account;
- there exist, at least in principle, methods of inquiry that if correctly pursued will yield such an account;
- methods of inquiry are to be accepted on the basis of whether they can yield such an account;
- individual theories and models in science are to be evaluated in large part on the basis of whether they provide (or come close to providing) a comprehensive and complete account based on fundamental principles.

Notice that the above description does not imply that the wanted complete theory of everything is necessarily unique. Nevertheless such the uniqueness assumption is often taken for granted.

The Vienna's Circle's thesis of the Unity of Science describes this unity in ontological terms. As Alan Richardson notes, when Rudolf Carnap claims to establish the unity of 'the object domain of science' he "does this by presenting a language in which all significant scientific discourse can be formulated. Putative metaphysical things such as essences, however, cannot be constructed — that is, they cannot be defined in the language — and this is the fact that Carnap uses to expunge metaphysical talk. Metaphysics does not speak of things in the object domain of science; there is only one such domain, and it contains all the objects that can be referred to, so metaphysics strictly does not speak of anything at all" [23, p. 6].

Carnap adds that "we can, of course, still differentiate various types of objects if they belong to different levels of the constructional system, or, in case they are on the same level, if their form of construction is different" [6, p. 9]. He gives an example of synthetic geometry where complex constructions are built from basic elements such as points, straight lines, and planes. Such constructions may involve several different layers but all statements about these constructions are ultimately the statements about their basic elements. So we have here different types of objects and yet a unified domain of objects from which they all arise.

The question arises: how big and how independent can be such complexes? It turns out that the "global" monism in the sense of the above definition allows, after all, for a pluralistic picture if one splits it into a number of "local" monisms based on independent complexes. A good example is a situation in today's non-classical logics to which we now turn.

2. Logical Pluralism and Logical Monism

The Tower of Babel is a cultural pattern, which recurs again and again. The first attempt of its erection, as it is well known, ended up with a catastrophe and produced multiple languages and the lack of understanding between the builders of this monster. However this was not the end of the story. A new Babel Tower dating back to Aristotle and the Stoics was the project of developing a unique and uniform logic supposed to provide rules of correct reasoning for all. This attempt seemed successful throughout the last two thousand years but eventually it failed as a result of the development and proliferation of the so-called non-classical logics. Some thinkers including Aristotle himself considered certain deviations from the Classical logic earlier but only in the beginning of the 20th century researches began to explore this new territory systematically. As a result many today's logicians hold a view according to which there exist many alternative systems of logic rather than a single "right" logic. This view is known under the name of *logical formalism.* Although the philosophical analysis of logical pluralism is still in its infancy the soundness of this view is hardly any longer questionable. It is possible that the logical pluralism will point to ways out of some deadends of modern logic and determine a strategy for developing logic in the 21st century. Implications of logical pluralism for the modern also still wait to be studied. In what follows we shall consider some problems of logical and metalogical pluralism and explore their implications for ontology and foundations of mathematics.

It may appear that the logical monism does not need an argumentative defense because it is supported by more then two thousand years of the history of logic. However the situation is not so is simple. Does the Classical logic in some sense imply the logical monism? Or perhaps some non-classical logic can play the same role of the only "right" logic common for all ? The Intuitionistic logic at certain point of history was considered as a candidate for this role. Later were considered some other candidates such as the Relevant logic, which allows one to avoid certain paradoxes appearing in the Classical logic. According to Stephen Read [21] the only purpose of logic is to distinguish between valid and invalid inferences. Hence, the argument goes, there is only one "true" logic, which can be nothing but the Relevant logic.

However if one takes into account how the concept of relevance has been modified in the course of the 20th century, one can hardly accept this and similar arguments of logical monists. All such arguments are ultimately ethical or aesthetic arguments rather than properly logical. They call for the "lost paradise", from where logics and logicians have been earlier expelled. The existing experience of metalogical researches indicates that there is no logical system satisfying all wanted metalogical properties and free from all paradoxes. As a matter of fact, it is difficult to single out even a short list of universal meta-properties which the ideal logical system of logical monists should necessarily possess.

Earlier R. Carnap [5] put forward the Principle of Tolerance in logic according to which logic should justify conclusions rather than establish some bans. There is no moral in logic and everyone has a liberty of building his or her own system of logic. As a matter of fact Carnap talks about the choice of formal language rather then the choice of logic. As it has been shown by G. Restall [22] one and the same language may admit for different logical consequence relations. So the distinction between language and logic is essential in this context.

J. Beall and G. Restall point to the following problem of logical pluralism:

"Which of these many logics governs your reasoning about how many logics there really are? In other words, which logic ought to govern your reasoning about the nature of logic itself? And indeed, which logic ought to govern your reasoning about the nature of logic itself?" [1, p. 6].

Indeed, a goal of logical pluralist is to study mutual relationships between the known logical systems. These logical systems can be seen either as a list of candidates for the same role of "the" unique "true" logic or as a friendly "logic community" providing different answers to the same questions. The builders of the Babel Tower eventually lost a common language and a mutual understanding. Does the existing logical community have same fate?

A basic problem of logical pluralism is the problem of relationships between different systems of logic. How such systems can be compared and estimated? If we recall that a logical theory is always a theory of some individual domain then the logical pluralism can be understood as the thesis according to which to one and the same domain, generally, admits for several alternative logics. Logical rules do not depend on empirical reliability, they cannot be cancelled because of empirical observations: logic is aprioristic by its very nature. Hartry Field argues [12] that a system of logic accepted a priori can be eventually replaced by an alternative logical system, equally designed a priori, under the pressure of facts. This view qualifies as a sort of *fallibilistic apriorism* (borrowing the term from the philosophy of science). However such a revision of logic can be possibly viewed as a mere recognition of the fact that the old logic simply did not correspond to the studied individual domain. As notices Ottavio Bueno [4] this possibility cannot be ruled out a priori.

3. Logical Eclecticism and Logical Relativism

The logical monism is a dogmatic position. The logical eclecticism, in its turn, is a variety of logical pluralism, which makes a choice of the best logical system from a list of such systems and aims at harmonizing competing approaches. On the other hand it operates like logical monism when it rejects certain moments of known logical systems as "erroneous".

A problem of logic eclecticism, as well as of any other sort of eclecticism, is the arbitrariness of choices: one chooses and uses certain principles without having any general theory justifying the choice. However the choice between logical systems becomes interesting when one translates problems formulated in some given logical framework into a different logical framework. This allows one to look at the given problem from a different viewpoint and sometimes helps to find an unexpected solution.

The same feature belongs to the position called *logical relativism.* Roy Cook describes it as follows: one qualifies as a relativist about a particular phenomenon if and only if one thinks that the correct account of it is a function of some distinct set of facts [9, p. 493]. How many similar correct accounts of the same set of facts can exist in principle? If the answer is that such accounts are multiple then this position reduces to a version of pluralism; of one assumes that there is only one such account then it reduces to monism. In this context Cook distinguishes between the *dependent* and *simple* varieties of pluralism. While former variety of pluralism is based on the relativism the latter is not. It may appear that an obvious example of the dependent pluralism is given by the *Tarskian Relativism* [31] according to which every term in a formal language can be equally treated either as logical or non-logical. But, as Varzi rightly notices, the Tarskian Relativism implies a stronger form of logical relativism according to which different ways of specifying the semantics of terms are equally admissible. It is possible, for example, that you and I agree that identity is a logical constant but you may think that it stands for a transitive relation whereas I may not accept this assumption.

4. Metalogical Relativism as the Consequence of Logical Pluralism

Varzi's paper referred to above makes it clear that Tarskian Relativism adds to the logical pluralism a new dimension related to the choice of logical semantics. Each variant of logical semantics comes with its own conception of logical consequence. Indeed, the usual definition of logical consequence - the conclusion follows from the given premises when in every case where the premises are true the consequence is also true - only looks neutral. In fact it involves the concept of truthfulness which depends on the chosen semantics of logical terms. Alternatively one may use in this definition a metaimplication opening thus yet a further dimension of pluralism.

Should be one's metalogic necessarily Classical? G. Priest, considering Tarski's theory of truth and his T-construction, writes that "sometimes it is said that Tarskian theory must be based on Classical logics: this logic is required for the construction to be performed. Such a claim is just plain false. It can be carried out in intuitionistic logics, paraconsistent logics, and, in fact, most logics" [20, p. 45].

Thus the Tarskian Relativism turns into the metalogical relativism and the metalogical pluralism. It allows for considering various alternative definitions of logical consequence such as: "the conclusion follows from premises if and only if any case in which each premise is true is also a case in which conclusion is relevantly true" (a case of relevant metalogic), "the conclusion follows from premises if and only if any case in which each premise is true is also a case in which conclusion is intuitionistically true" (a case of intuitionistic metalogics), "the conclusion follows from premises if and only if any case in which each premise is true is also a case in which conclusion is paraconsistently true" (a case of paraconsistent metalogic), "the conclusion follows from premises if and only if any case in which each premise is true is also a case in which conclusion is quantum logically true" (a case of quantum metalogic), etc.

Moreover, apparently nothing prevents one from correlating one's concept of logical consequence with a non-Classical logic. Then the above definition can be modified as follows: "the conclusion intuitionistically follows from premises if and only if any case in which each premise is intuitionistically true is also a case in which conclusion is intuitionistically true" (the case of intuitionistic logic and metalogic), "the conclusion relevantly follows from premises if and only if any case in which each premise is relevantly true is also a case in which conclusion is relevantly true" (the case of relevant logic and metalogic), "the conclusion intuitionistically follows from premises if and only if any case in which each premise is relevantly true is also a case in which conclusion is relevantly true" (the case of relevant metalogic for Intuitionistic logic), "the conclusion relevantly follows from premises if and only if any case in which each premise is intuitionistically true is also a case in which conclusion is intuitionistically true" (the case of relevant metalogic for intuitionistic logic), etc. Here the choice may be limited by certain specific properties of these 'cases' [24, p. 396].

Thus we can formulate a "metalogical" definition of logical consequence as follows:

A1onclusion is valid in the given logic if in the corresponding metalogic the validity of premises implies the validity of the conclusion.

On this basis it is possible to construe two further different versions of the above metalogical definition:

(i) a conclusion is valid in some logic if in some metalogic the validity of premises implies the validity of the conclusion.

(ii) a conclusion is valid in some logic if in all metalogics the validity of premises implies the validity of the conclusion.

The second version is hardly realistic since all possible metalogics can be hardly taken into account. One may also suspect that the choice of metalogic may depend on the existence of 'translation' from certain logic to the given logic. Indeed, all "mixed" principles arise via a meddling or substituting semantics of one logic to another. These semantic operations may provide grounds for further arguments pro or contra the monistic (when logic always coincides with the metalogic) and (when logic and metalogic may differ conceptually) points of view.

A non-Classical metaimplication gives rise to a meta-metalogical definition of logical consequence as follows:

- A conclusion follows from premises iff the truth of the conclusion follows from the truth of premises iff in all cases the truth of premises implies the truth of the conclusion.

On the one hand, this is a bad infinity. But on the other hand, this obtained situation can be described in terms of S. Kripke's theory of truth [16]:

- A conclusion logically follows from premises if and only if the truth of the conclusion follows from the truth of premises if and only if the truth of the truth of the conclusion follows from the truth of the truth of premises.
- *Mutatis mutandis* in case of the 'mixed' principle. In this case in addition to Kripke's considerations of cases of the truth or falsity at corresponding meta-levels we need also to construe the truth on pluralistic variants of meta-levels.

5. Logical Pluralism and Universal Logics

How statements of the form 'A follows from B iff B is true implies A is true in metalogic **M**' can be compared in the case of different metalogics? Some authors suggest that this can be done with a theory of *Universal Logic* that would provided criteria for such a comparison (see [33],[35]).The Universal Logic (UL) is a theory of translatability and combination of logical systems. The above statements can be compared with UL as follows. First one constructs a translation F from (meta)logic Y_1 to (meta)logic Y_2. Then

'A follows from B iff B *is true* implies in Y_1 A *is true*'

translates under F into

'A follows from B iff $F(B$ *is true*$)$ implies in Y_2 $F(A$ *is true*$)$'.

If such translations between different metalogics exist then we can speak about a local metalogical monism: the translatability gives us an invariant kernel preserved through translations.

Instead of linking by means of translation we would consider, using methods of universal logic, the combinations of two formulations, e.g. join of two formulations. In this case join of two logics gives us the uniform logic possessing properties of both initial logics. In particular, in union $Y_1 \oplus Y_2$ of two metalogics Y_1 and Y_2 "joint" consequence relation is defined by means of a condition:

if from A *is true* in one metalogic (Y_1 or Y_2) follows B *is true* in the same metalogic then from A *jointly* follows (i.e. within the framework of the metalogic $Y_1 \oplus Y_2$) B.

To put it more precisely "jointly follows" gives us that

- A follows from B iff B *is true* implies in $Y_1 \oplus Y_2$ A *is true*

Instead of unions of metalogics one can also use their product (taking pairs of metaformulas as new metaformulas), so the definition becomes

if from A *multiplicatively* follows (i.e. within the framework of the metalogic $Y_1 \otimes Y_2$) B then from A *is true* in both metalogics (Y_1 and Y_2) follows B *is true*.

"Multiplicativeness" gives us that

- if A *is true* follows in metalogic $Y_1 \otimes Y_2$ from B *is true* then A follows from B.

Similarly one can consider the *exponential* and *co-exponential* local metalogical monism combining metasystemsY_1, Y_2 into $Y_1 \Rightarrow Y_2$ and $Y_1 \Leftarrow Y_2$ respectively and then use the "implications" of these combined metasystems in the definition of logical consequence of the same form (provided such combinations as allowed in UL).

An obstacle for this project is the omniscience problem: we cannot explicitly describe all possible logics in advance and hence cannot accomplish all possible combinations of logics. The above types of combinations of (meta)logical systems do not exhaust all possible combinations being only the most universal ones.

6. From Logical to Ontological Pluralism

According to J. Bocheński, the modern logic is "a most abstract theory of objects whatsoever" or a "physics of the object in general". Thus "logic, as it is now constituted, has a subject matter similar to that of ontology" [3, p. 288]. In effect, ontology is a prolegomenon to logic. While ontology is an informal, intuitive inquiry into the basic properties and basic aspects of entities in general, logic is the systematic, formal, axiomatic elaboration of these ontological intuition. While ontology as it is usually practiced is the most abstract theory of real entities, logic in its present state is the general ontology of both real and ideal entities [3, p. 290].

Thereby logical pluralism is 'dangerous' because it implies the ontological pluralism. Since any logical theory is always a theory of some domain of individuals, the acceptance of this or that logic compels to certain assumptions, hypotheses about the cognizable objects inhabiting this area and described by our theory. It is a good thing if we are in a position to control these assumptions; too often such assumptions remain tacit.

Ontological assumptions are specific to languages - artificial or natural. The term "ontological commitment" that denotes this phenomenon can be understood either as an ontological assumption, or an ontological obligation or as an ontological hypothesis. Scientific artificial languages, which are always designed for a definite purpose, may enforce certain ontological commitments not intended by their designers.

Such troubles are rooted in the fact that formal languages designed for the scientific purposes should cope with two different ontologies, one of which represents the domain of scientific inquiry while the other belongs to the language itself and depends on its formal properties. The history of science of the XXth century makes it clear that interactions between these two ontological layers cannot be ignored.

How ontological assumptions of a given formal language can be identified? An answer is given by A. Church's criterion: a language carries an ontological commitment associated with every sentence, which is analytic in this language, i.e., of every true sentence whose truth is granted by the semantics of this language. The distinction between analytical and synthetic sentences is made here as follows: "One can single out two types of propositions: propositions, whose truth or falsity should be established on the basis of semantic rules of the system, and propositions, whose truth or falsity cannot not be seen from them. Such division of statements of language in respect to fixed semantic system, division on analytical and synthetic in this sense, in our opinion, is indisputable. The question consists in their exact definition and interpretation" [26, p. 88].

The usual semantics of the first-order classical logic is given in terms of its Tarskian models. The *universe of all sets* and the related set theory provide in this case the proper ontology for this language. Thus in the case of this particular language the 'theory of objects in general' coincides with some version of set theory (possibly with urelements and empirical predicates, see [8]).

However the set theory is itself an elementary theory, i.e., a set of formal statements deduced from a conservative axiomatic extension of predicate logic with certain non-logical axioms, which describe formal properties of predicate $\in$. By modifying the logical part of this theory one can obtain a new theory based on some non-classical logic: paraconsistent, relevant, quantum, fuzzy etc. Thus one obtains a class of non-classical set-theoretic universes associated with their non-classical underlying logics.

There is another simple argument supporting the claim that logical pluralism implies the pluralism of universes. Consider usual definitions of operations of join $\cup$, meet $\cap$ and complementation / on sets

$x \cup y =_{def} \{a : a \in x \vee a \in y\}$,

$x \cap y =_{def} \{a : a \in x \wedge a \in y\}$,

$x/y =_{def} \{a : a \in x \wedge \neg(a \in y)\}$.

A pluralist may ask: what type of connectives $\vee$ (or), $\wedge$ (and), $\neg$ (it is incorrect, that) are used in these definitions? If these are classical connectives then the algebra of subsets of a given set is Boolean.

But what happens, if one modifies the operations on sets using non-classical logic connectives $\vee$, $\wedge$, $\neg$ and then construes an algebra for the obtained new operations? Since in Tarskian models set-theoretic operations are responsible for truth values of formulas this provides us with an interpretation of a non-classical logic in the Classical universe. In this way one can interpret in the given Classical universe as many non-Classical logics as one wants. One can also use a non-Classical universe and introduce in it Classical set-theoretic operations. So one gets an interpretation of Classical logic (along with non-Calssical ones) in a non-Classical universe.

Is there a way to check whether "our" universe is Classical or non-Classical? Logical pluralism gives an answer in negative. One can assume the existence of a global underlying logic for a given universe but this global logic does not determine any set of local logics, which this universe may admit. Of course, we talk about global and local logics in this context only metaphorically as markers fixing a state of affairs.

7. Non-Classical Mathematics: as Many Logics as Mathematics

The 20th century has witnessed how the original intuitionist and constructivist renderings of set theory, arithmetic, analysis, etc. were later accompanied by those based on relevant, paraconsistent, non-contractive, modal, and other non-Classical logical frameworks. This development led to the ongoing scientific program of "Non-Classical Mathematics". At the conference "Non-Classical Mathematics 2009" (June 2009, Hejnice, Czech Republic) the Non-Classical Mathematics 2009 has been defined as a study of mathematics which is formalized by means of non-Classical logics. The Program of this conference included the following sections:

- Intuitionistic mathematics: Heyting arithmetic, intuitionistic set theory, topos-theoretic foundations of mathematics;
- Constructive mathematics: constructive set or type theories, pointless topology;
- Substructural mathematics: relevant arithmetic, non-contractive naive set theories, axiomatic fuzzy set theories;
- Inconsistent mathematics: calculi of infinitesimals, inconsistent set theories;
- Modal mathematics: arithmetic or set theory with epistemic, alethic, or other

modalities, modal comprehension principles, modal treatment of vague objects, modal structuralism.

It is obvious, that there is not one but many true mathematics. But it remains unclear how these different mathematics interact. Are they complementary or mutually exclusive? This situation resembles that with non-Euclidean geometries. This analogy suggests questions like this: is our mathematics globally Classical, and only locally non-Classical or, on the contrary, it is globally non-Classical and locally Classical?

G. Takeuti develops a *quantum set theory*, which involves a quantum-valued universe. It remains however unclear whether the "mathematics based on quantum logic has a very rich mathematical content. This is clearly shown by the fact that there are many complete Boolean algebras inside quantum logic. For each complete Boolean algebra $\mathcal{B}$, mathematics based on $\mathcal{B}$ has been shown by our work on Boolean valued analysis to have rich mathematical meaning. Since mathematics based on $\mathcal{B}$ can be considered as a sub-theory of mathematics based on quantum logic, there is no doubt about the fact that mathematics based on quantum logic is very rich. The situation seems to be the following. Mathematics based on quantum logic is too gigantic to see through clearly" [29, p. 303].

R. Meyer proposes a construction of relevant arithmetic built along the same 'pluralistic' line on a basis of relevant logic [18]. Recall that Peano Arithmetic (PA) is based on the first-order Classical logic (FOL) and involves a number of non-logical axioms. Relevant Peano arithmetic R# according to Meyer is obtained from PA via a replacement of FOL by a system of relevant logic R, leaving the non-logical axioms unchanged.

One more instance of a non-Classical mathematical theory is given by K. Mortensen in his book 'Inconsistent Mathematics' [19]. Claiming that "philosophers have hitherto attempted to understand the nature of contradiction, the point however is to change it", Mortensen describes the mathematics based on the paraconsistent logic.

In a more sophisticated way a non-Classical logical basis is used in theories of formal topology. A topological structure is usually specified via a specification of set of opens closed under the set-theoretical intersection. By modifying the concept of intersection one obtains a family of new topologies. In particular the set-theoretic intersection can be replaced by the operation of monoidal multiplication. Such constructions can be made with a non-Classical set theory interpreted in a Classical universe.

When one accepts logical pluralism and allows for various logical foundations formal topological properties can be equally taken into account. An example of such an account can be found in the Quantum theory (QT). G. Birkhoff and J. von Neumann demonstrated an equivalence between experimental statements of QT and subspaces of Hilbert spaces. The set-theoretic intersection of two given experimental statements (represented as the closed vector subspaces of Hilbert space) is also an experimental statement (i.e., a closed vector subspace of Hilbert space). Whence one easily defines a topological structure using the standard definition of boundary.

However when one takes into account the fact that the negation of an experimental statement is its orthogonal complementation, one obtains a formal topology, which differs from its Classical counterpart.

Today's mathematics is going trough a paradigm shift in its foundations from the set-theoretic paradigm to the category-theoretic one. From a logical point of view Category theory like Set theory is an elementary theory based on the Classical first-order calculus with equality.

Following N.C.A. da Costa, O. Bueno and A. Volkov [10] one can build *paraconsistent* elementary theory of categories using the paraconsistent logic $C_1^=$. The axioms of the paraconsistent category theory include all usual axioms with the Classical negation and some new axioms with the paraconsistent negation. One can also construct a paraconsistent category theory [35] using axioms for category theory proposed by G. Blanc and M.-R. Donnadieu [2].

Recall that *topos* is a category of a special kind in which there exists a special object bearing a structure of Heyting algebra. The above algorithm for developing non-Classical mathematical theories allows one to build various 'quasi-toposes' by replacing Heyting algebra with some other algebras of logics. For example, the replacement of Heyting algebra by the paraconsistent da Costa algebra brings a 'potos' (aka da Costa topos). A potos is a paraconsistent universe in which one can develop paraconsistent mathematical theories just as in the case of the intuitionistic mathematics. While in the usual topos the paraconsistency features only in special constructions and in this sense remain local artefacts, in a potos the paraconsistency is organic and underlies all further constructions. In the paraconsistent universe the Classical mathematics features as an artefact, i.e. as a local deviation from the paraconsistent regularities.

Similarly one can replace Heyting algebra with the relevant one and thus obtain a category called 'reltos' which interprets the relevant logic and allows for developing the relevant mathematics [36]. This short list does not exhaust all possibilities for developing the non-Classical mathematics.

Toposes, generally, are non-Classical constructions, namely, constructive intuitionistic universes. “By imposing natural conditions on a topos (extensionality, sections for epics, natural numbers object), we can make it correspond precisely to a model of Classical set theory. Thus, to the extent that set theory provides a foundation for mathematics, so too does topos theory” [13, p. 344]. What a “natural condition” means precisely in this context?

In a topos-theoretic context a Classical universe is a local construction (being a special case of general topos) while the nature of general topos is purely intuitionistic, i.e., essentially non-Classical. Thus the general topos serves as a global non-Classical foundation of mathematics, which can be Classical locally.

Other kinds of non-Classical can be similarly obtained locally in the same global intuitionistic context. This can be achieved with Lawvere's 'variable sets' aka intensional sets aka “set-theoretical concepts” (R. Goldblatt's terminology). According to Goldblatt the intension or meaning of a given expression, is an “individual concept expressed by it”. For example, if $\varphi(x)$ is the statement 'x is a finite ordinal' then the intension of φ is the *concept* of a finite ordinal. In the categorical language this concept is represented by a functor that assigns to each $p \in P$ a set of things known “at stage p” to be finite ordinals [13, p. 212].

By varying p, one can impose different “natural restrictions” on given sets of individuals and thus obtain set-theoretic concepts, which describe non-Classical sets. In particular, such a variation can be used for interpreting quantum logics in toposes; in this case the obtained set-theoretic concepts characterize quantum sets.

Likewise it is possible to use functor category Set^A from the so-called CN-category (which is a category-theoretic equivalent of da Costa algebra) to category Set. This category is a topos. Notice that the completeness of da Costa C^1 paraconsistent system has been proved with respect to a similar topos [32]. A similar approach can be used in the case of relevant logic R [34].

Presently only a small minority of mathematicians expresses an interest in the non-Classical mathematics (beyond its intuitionistic and constructive varieties, which are related to the theory of computability). There are two reasons for this. First, the non-Classical mathematics so far did not bring anything interesting for the viewpoint of mathematical novelty. Researches in this field still focus on mathematical characteristics of non-classical logics and their models. This common tendency is evident in spite of some noticeable exceptions (e.g. Kris Mortensen's book 'Inconsistent Mathematics', an attempt by K. Piron to reformulate quantum mechanics on quantum logic foundations). Perhaps the development of interactive non-Classical provers and decision-making systems will be able to make this research filled more vivid. The effectiveness and the convenience of the human-machine interaction may serve as a strong argument in favour of this or that non-Classical mathematics.

Second, there is a danger for non-classical mathematician to become a 'hero of deserted landscapes'. Polish science-fiction writer Stanisław Lem distinguished between three kinds of genius [17, p. 89]. A genius of the third kind is an ordinary genius who is beyond the intellectual scope of his age. A genius of the second kind is a hard nut, which his contemporaries cannot crack. Such a genius usually gets a postmortem recognition. Geniuses of the first and the highest kind remain wholly unknown - both during their lifetimes and after their deaths. Their intellectual impact is so revolutionary that no one can evaluate it. Lem provides a fictitious historical example of a manuscript by an anonymous Florentian mathematician of XVIII century, which *prima facie* appeared to be a work in Alchemistry but at a closer examination turned out to be a project of alternative mathematics, which differed drastically from our mathematics as we know it. Checking whether this alternative mathematics is better or worse than the usual one would require a lifetime work of hundreds of scientists working on the manuscript by the Florence Anonymous in a way similar to which Bolyai, Lobachevsky and Riemann worked on Euclid. In reality most mathematicians simply avoid developing any 'parallel' mathematics.

8. <u>Conclusion</u>

Recent developments in logic support a pluralistic logical picture of the world. Besides, it should not be supposed that such situation is true only for logics. The emergence of non-Classical mathematics should not be seen as a supporting evidence for logicism. It should be rather understood as a natural consequence of the internal pluralism of logic which has been made explicit in recent developments. Having in mind D. Hilbert's view according to which logic is a metamathematics one can see that logical pluralism implies the plurality of mathematics, i.e., the plurality of mathematical pictures of the world.

Describing the Classical science Kant famously remarked that "each science is as much a science as much there is mathematics in it". Can one really expect a 'pluralization' of such scientific disciplines as physics and biology along with the pluralization of mathematics? From the Classical point of view the answer should be affirmative. However, we are living in the epoch of post-non-Classical rather than Classical science. For this reason scientific pluralism is limited with a variety of systems of social values and goals, which dictate choices of our research strategies. According to V.S. Stepin "The post-non-classical type of scientific rationality broadens the field of reflection over activity. It takes into account correlation of obtained knowledge of the object not only with specificity of means and operations of activity, but also with value-goal structures. Here we explicate the connection between intrascience goals and extra-scientific, social values and goals" [27, p. 634]. So the pluralism of the modern logic is rather a precondition of freedom in our choices of logical toolkits, which determines directions of our researches.

The development of logic in the 20-th century made clear that certain metalogical characteristics which were earlier believed to be universal were actually not universal. This concerns, in particular, the completeness and the consistency of logical systems, which make no sense in the case of paraconsistent logical systems (albeit they have such properties as paraconsistency and paracompleteness). Notice that relevant logics can be paraconsistent and at the same time consistent and complete. Such facts provide an additional evidence in favor of the post-non-Classical view according to which a logician or a mathematician should select his or her formal toolkit on the basis of certain goals, values and norms.

References

[1] Beall J.C. and Restall G. Defending Logical Pluralism. In: John Woods and Bryson Brown (eds.). *Logical Consequence: Rival Approaches. Proceedings of the 1999 Conference of the Society of Exact Philosophy*, Stanmore: Hermes, 2001. P. 1-22.

[2] Blanc G., Donnadieu M. R. Axiomatisation de la categorie des categories. *Cah. Topol. Geom. Different.* XVII, 2, 1976. P. 1-38.

[3] Bocheński J.M. Logic and Ontology. *Philosophy East and West* 24, VII(3), 1974. P. 275-292.

[4] Bueno O. Is Logic A Priori? *The Harvard Review of Philosophy*, vol. XVII, 2010. P. 105-117.

[5] Carnap R. The Logical Syntax of Language. Littlefield, Adams and Co., 1959. Translated by Amethe Smeaton.

[6] Carnap R. The Logical Structure of the World and Pseudoproblems in Philosophy. Chicago and La Salle, Illinois: Open Court Publishing, 2005.

[7] Cartwright N. The Dappled World. Cambridge: Cambridge University Press, 1999.

[8] Cochiarella N.B. Predication Versus Membership in the Distinction between Logic as Language and Logic as Calculus. *Synthese* 77, 1988. P. 37-72.

[9] Cook R.T. Let a thousand flowers bloom: A tour of logical pluralism. *Philosophy Compass*, 5(6), 2010. P. 492–504.

[10] Da Costa N.C.A., Bueno O., Volkov A. Outline of a Paraconsistent Category Theory. In: Weingartner P. (ed.). *Alternative Logics. Do Science Need Them?* Springer, Berlin, Heidelberg, New York, 2004. P. 95-114.

[11] Dupré J. The Disorder of Things: Metaphysical Foundations of the Disunity of Science. Cambridge, Mass.: Harvard University Press, 1993.

[12] Field H. Epistemological Nonfactualism and the A Prioricity of Logic. *Philosophical Studies* 92, 1998. P. 1–24.

[13] Goldblatt R. Topoi. The Categorial Analysis of Logic. Amsterdam-New York-Oxford: North-Holland, 1984.

[14] Keller S.H., Longino H.E. and Waters C.K. Introduction: The Pluralist Stance. In: Keller S.H., Longino H.E. and Waters C.K. (eds.). *Scientific Pluralism (Minnesota studies in the philosophy of science; 19).* Minneapolis: University of Minnesota Press, 2006. P. vii–xxix.

[15] Kraft V. Der Wiener Kreis. Der Ursprung des Neopositivismus, Wien-New York: Springer-Verlag, 1968.

[16] Kripke S. (1975). Outline of Truth Theory. *The Journal of Philosophy*, 72(9), 695-717.

[17] Lem S. Doskonała próżnia. Wiełkość urojona. Krakow: Wydawnictwo Literackie, 1974.

[18] Meyer R.K.. Relevant arithmetic. *Bulletin of the Section of Logic*, 5:133–7, 1976.

[19] Mortensen K. Inconsistent Mathematics. Dordrecht: Kluwer, 1995.

[20] Priest G. Doubt Truth to be a Liar. Oxford: Clarendon Press, 2008.

[21] Read S. Monism: the one true logic. In: D. DeVidi and T. Kenyon (eds.). *A Logical Approach to Philosophy: Essays in Honour of Graham Solomon*, Springer, 2006. P. 193-209.

[22] Restall G. Carnap's Tolerance, Meaning and Logical Pluralism. *Journal of Philosophy* 99, 2002. P. 426-443.

[23] Richardson A.W. The Many Unities of Science: Politics, Semantics, and Ontology. In: *Scientific Pluralism (Minnesota studies in the philosophy of science; 19)*. Minneapolis: University of Minnesota Press, 2006. P. 1-25.

[24] Routley R., Meyer R. (1973). Semantics of Entailment. In: H. Leblanc (ed.), *Truth, Syntax and Modality* (199-243). Amsterdam-London.

[25] Smirnov V.A.. Logicheskiye metody analiza nauchnogo znaniya (Logical Methods of Scientific Knowledge Analysis). Moscow:Nauka, 1987. [in Russian]

[26] Smirnova E.D. Analiticheskaya istinnost' (An Analytical Truth). *Metodologicheskiye aspekty kognitivnykh protsessov (Vychislitel'nye sistemy, 172)*. Novosibirsk, 2002. P. 74-134. [in Russian]

[27] Stepin V.S. Theoretical Knowledge. Synthese Library / volume 326. Springer, 2005.

[28] Suppes P. The Plurality of Science. In: Peter Asquith and Ian Hacking (eds.). *PSA 1978: Proceedings of the 1978 Biennial Meeting of the Philosophy of Science Association*, vol. 2. East Lansing, Mich.: Philosophy of Science Association, 1978. P. 3–16.

[29] Takeuti G. Quantum Set Theory. In: Beltrametti S., Fraassen B. Van (eds.). *Current Issues on quantum logic*. New York; London: Plenum, 1981. P. 303-322.

[30] Takeuti G. and Titani S. Fuzzy Logic and fuzzy set theory. *Arch. Math. Log.*, 1992. P. 1-32.

[31] Varzi A.C. On Logical Relativity. *Philosophical Issues* 10, 2002. P. 197–219.

[32] Vasyukov V.L. Paraconsistency in Categories. In: D. Batens, C. Mortensen, G. Priest and J.-P. van Bendegem (eds.), *Frontiers of Paraconsistent Logic*. Research Studies Press Ltd., Baldock, Hartfordshire, England, 2000. P. 263-278.

[33] Vasyukov V.L. Structuring the Universe of Universal Logic. *Logica Universalis*, vol.1, 2, 2007. P. 277-294.

[34] Vasyukov V.L. Paraconsistency in Categories: Case of Relevant Logic. *Studia Logica*, Volume 98, Issue 3, 2011. P. 429-443.

[35] Vasyukov V.L. Logicheskiy pluralism i neklassicheskaya teoriya kategoriy (Logical Pluralism and Non-Classical Category Theory). *Logicheskiye Issledovaniya*, issue 18, Moscow-St.Petersburg: Tsentr gumanitarnych initsiativ, 2012. P. 60-76. [in Russian]

[36] Vasyukov V.L. Reltoses Semantics for Relevant Logic. In: *Devyatye Smirnovskiye chteniya po logike. Materialy mezhdunarodnoy nauchnoy konferentsiyi 17-19 iyunia 2015 g.* Moscow: Sovremennye tetradi. 2015. P. 13-15.