Models of HoTT and the Semantic View of Theories

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Tarski-style Model theory

$\mathit{Model} \models \mathit{Theory}$
in words: The Model *satisfies* (= makes true) the corresponding Theory.

$\mathit{Interpretation} : \mathit{Signature} \to \mathit{Structure}$

Assumptions 1-2

- A theory is a system of formal sentences, which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms.

Two distinct points of a straight line completely determine that line

If different points A,B belong to straight line a and to straight line b then a is identical to b.

Assumption 3

Structures are *set-theoretic* structures.

Tarski 1941

"For precision it may be added, that the considerations which we sketched here are applicable to any deductive theory in whose construction logic is presupposed, but their application to logic itself brings about certain complications which we would rather not discuss here."

Compare Tarski's topological semantics for Classical and Intuitionistic propositional calculi (1935)

Lawvere 1963: Functorial Semantics of Algebraic Theories

Idea: use categories instead of signatures (thus blurring the distinction btw. logical and non-logical terms)

Algebraic (Lawvere) theory: category LT with finite products and distinguished object X s.th. every object A in C is isomorphic to X^n for some finite number n.

Model: $LT \rightarrow SET$ that preserves finite limits.
Generalized Models: $LT \rightarrow C$ where C has finite limits.

Sketches

Observation: Even if (small) category C does not have (co)limits the presheaf category $\hat{C} = [C, SET]$ does. This allows for using *sketches* "instead of" theories.

Theories in the Categorical perspective (after Awodey & Bauer)

$$\textit{Theory} \rightarrow \textit{Category}$$

- cartesian theories (only $\wedge$ and $\top$)
- regular theories (only $\wedge$ and $\top$ and $\exists$)
  (*regular* category: finite completeness plus image factorization stable under pullbacks)
- coherent theories (plus $\vee$ and $\bot$)
  (*coherent* category: regularity plus unions stable under base change)
- geometric theories (plus infinitary $\bigvee$)
  (*geometric* category: infinitary coherent)

Syntactic aka Classifying aka Walking Categories

Idea: a category "freely generated from the syntax"

- Lawvere's theory

- contextual category (contexts as objects and substitutions as morphisms)

Generic Models

Universal property: $Synt(T)$ is initial in $Mod(T) = [T, C]$

Internal Language

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

$$Model : T \rightarrow Lang(C)$$

(in Theories)

Problem:

It is not clear whether Tarski's notion of model based on the satisfaction relation and his T-schema covers the functorial notion(s) of model in all cases. Categorical model theory may need an independent philosophical underpinning.

Claim:

Existing models of Homotopy Type theory are <u>not</u> Tarskian models and cannot be described in terms of the satisfaction relation and the T-schema.

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

Martin-Löf 1983

"Classical" notion of proposition as truth-value is rejected and replaced by the "intuitionistic" one:

"A proposition is defined by laying down what counts as a proof of the proposition."

"A proposition is true if it has a proof, that is , if a proof of it can be given."

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T

MLTT: Proposition (M-L 1983)

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h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-hierarchy

(-2) single point pt;

(-1) the empty space $\emptyset$ and the point pt : truth values aka *classical* or "mere" propositions

(0) sets aka *intuitionistic* propositions aka theorems

(1) (flat) groupoids

(2) 2-groupoids

-
-

(n) n-groupoids

- ...

(ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does not take this structure into account. HoTT semantics does.

HoTT semantics (or the version thereof that I defend) does not license the idea that *every* type is a proposition. It recovers within

the MLTT syntax the classical notion of proposition as well as the intuitionistic notion of proposition-as-set (under a different name) and determines the precise place of both in the hierarchy of types. These semantic decisions are not arbitrary but based on the robust mathematical structure of *h*-stratification of types. *h*-stratification should be reflected semantically. Logical rules are specializations of more general constructive rules.

HoTT semantics for $t : T$ for (-1)-types

propositions and truth-values

HoTT semantics for $t : T$ for (0)-types

theorems and their proofs / sets and their elements

HoTT semantics for $t : T$ for higher -types

(also valid for lower types):

spaces and points, which support higher-order structures from elements of some other spaces (viz. map spaces);

objects are points;
constructions are points provided with additional higher-order structures: paths, surfaces (homotopies), etc.

Competing approaches to modeling HoTT

- Awodey: classifying categories, natural models
- Voevodsky: contextual categories (Cartmell), C - systems, Initiality Conjecture (open)

Since HoTT is *not* a system of sentences (propositions) Tarski's notion of model may account at most for the propositional fragment (level) of HoTT/MLTT

Theories, which are *not* systems of sentences are less exotic than one could think. Gentzen's Natural Deduction and the geometrical theory of Euclid's *Elements*, Books 1-4 are other examples.

Euclid's *Common Notions* and *Postulates* are rules rather than axioms in the modern sense of the term.

Arguably a typical scientific theory is not a system of propositions either (The "Non-Statement View of Theories" of P. Suppes, B. van Fraassen et al.).

(Once) Received view

A theory is a set of propositions (expressed in a formal language L). It can be possibly (but not generally) represented through a list of *axioms* and described as the deductive closure of these axioms (Carnap).

Non-statement view

A theory is a class of models but not an axiom system, nor its deductive closure. (Suppes, Sneed, Stegmüller, Balzer, Moulines, van Fraassen)
Suppes 2002: Term "model" is used in logic and science similarly.
One and the same theory may allow for many different axiomatizations.

Bourbaki-style representation of theories

is arguably useful for logical analysis but hardly useful for general research and educational purposes.

Bunge 1972

In his epoch-making book [von Neumann, *Mathematical Foundations of Quantum Mechanics*, 1932]), which enriched the mathematical framework of the theory, von Neumann is wrongly supposed to have laid down the axiomatic foundations of quantum mechanics. As a matter of fact his exposition lacks all the characteristics of modern axiomatics [. . .]. Yet for some strange reason it passes for a model of physical axiomatics. [. . .] In the meantime the study of axiom systems made dramatic strides.

Comment

There is a huge gap between logicians' and physicists' concepts of theories and, more specifically, axiomatic theories. The idea of logical and metaphysical foundations of natural science, has been banished from Physics in the early 17th century by Galileo and his followers as a part of the traditional Aristotelian background of their contemporary physics. Logic remains under suspicion among physicists ever since.

Bunge 1972

There is a single theory that starts from scratch: mathematical logic (which is actually a set of theories). Indeed, the truths of logic or tautologies [. . .] are those that can be proved without resorting to assumptions other than the rules of logic. All other theories presuppose at least logic and usually a lot more. More precisely, the least a mathematical or a scientific theory takes for granted is the so-called ordinary (two-valued) predicate calculus enriched with the microtheory of identity.

Comment

In Bunge's view logic is (1) fixed and (2) self-explanatory. This view on logic and its role in axiomatic theories is also Hilbert's both in 1899 (Foundations of Geometry) and in 1934-38 (Foundations of Mathematics).

Putnam in "Is Logic Empirical?" (1968) rejects this neo-Aristotelian (modulo an update of logical calculus) view but does not propose any alternative axiomatic architecture for scientific theories.

Hans Halvorson

"Scientific Theories" (forthcoming). The controversy between syntactic and semantics approaches is artificial. There is a duality between syntax and semantics (Lawvere 1963, Awodey&Forssell 2013, nlab). The Bourbaki-style semantic presentation is language-dependent and gives little or no advantage. The "semantic view" arguments is are no longer relevant in the presence of <u>formal</u> theories of semantics.

Indeed, the semantic view can be understood as an attempt to pave the wide gap between logicians' and physicists' theory concepts. Unlike Halvorson I do not believe that the gap is paved by the standard theories of formal semantics. However think that a HoTT-based theory of semantics may help to solve the problem (more shortly).

Hans Halvorson

Claim [in support of the semantics view] : Scientists often deal with collections of models that are not elementary classes, i.e. aren't the collection of models of some set of first-order sentences.

Hans Halvorson: comment

This claim is strange, for it seems to indicate that scientists work with classes of L-structures (for some language L) that are not elementary classes (i.e. not the classes of models of a set of first-order sentences). I happen to know of no such example. Certainly, scientists work with classes of models that are not in any obvious sense elementary classes, but largely because they haven't been given a precise mathematical definition.

In my view the above claim is justified. The notion of model as a truth-maker does not fully characterize a typical scientific model.

E.g. the model of Solar System of Newton's *Principia* (evidently makes true certain sentences but also) comprises a non-propositional structure, which allows one to synthesize planets' curvilinear orbits from their supposed elementary (infinitesimal) rectilinear motions (and, dually, analyze the curvilinear orbits into infinitesimal rectilinear elements). There is no straightforward way to represent such a structure via a logical structure (albeit a roundabout way to this may exist). Moreover, for epistemological reasons such a translation can be undesirable.

Desiderata for a formal framework for a scientific theory

- support deduction from first principles (first elements), including non-propositional ones (primitive objects, types, etc.)
- combine logical rules with constructive rules (i.e., rules for non-propositional objects)
- support thought-experimentation

Constructive axiomatic method

Theories satisfying the above desiderata I shall call *constructive* axiomatic theories.

This use of the term “constructive” has a historical grounding (ex. Hilbert&Bernays 1934) but is not standard. This notion of being constructive does not fix any specific set of rules.

Examples of constructive axiomatic theories

- Euclid's geometry
- HoTT

"Low-level" scientific models are concrete *methods* for conducting particular experiments (experimental design) and making observations.

Even if the sole purpose of experiments and observation is to give yes-no answers to certain questions an experiment and an observation need to be *designed*.

Conclusion 1

The critique of the standard formal semantics by proponents of the "semantic view of scientific theories" points to a real problem. In particular, this problem concerns the standard *logical* semantics. The standard notion of theory is a set of sentences structured by the relation of inference does not account to what people want to call by this name in science. A scientific theory is rather a system of rules for model building. Such a system of rules does not, generally, reduces to a system of rules for operating only with sentences.

Conclusion 2

HoTT is a formal mathematical framework capable to support a model-oriented experimental reasoning in science. The concept(s) of *model* developed with HoTT differ(s) from the standard Tarski's concept and deserve a further epistemological study. There are reasons to think that such (a) novel concept(s) of model may better serve in science than Tarski's concept. This makes HoTT a strong candidate for the role of formal setting for the axiomatic Physics and other axiomatic scientific theories.

СПАСИБО!