Rules vs. Axioms

Two Axiomatic Styles from a Modern Perspective

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12 May 2017

Assumptions behind Hilbert's Axiomatic Method

- A theory is a system of formal sentences, which are satisfied in a model (including the intended model). An axiomatization of theory T amounts to a choice of an appropriate subset A of T-sentences such that all T-sentences can be inferred from A-sentences (aka *axioms* of T) according to certain fixed rules of inference, see the next item.
- In a formal theory logical and non-logical terms are distinct. Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms. Rules (of inference) belong to the logical part (whether the inference is understood syntactically or semantically).

Two distinct points of a straight line
completely determine that line

Epistemological Grounding

Theoretical reasoning reduces to logical reasoning, i.e., reasoning according the rules and standards of logic. All extra-logical contents of a given theory are made explicit via (non-logical) axioms and their interpretations.

Tarski's additional assumption

Structures are *set-theoretic* structures.

Axiomatization of logic(al theories): a special case?

Hilbert & Ackermann 1928: YES

Problem (Hintikka et al.): the epistemological grounding as above does not apply

Reply to Hintikka: the distinction between logical and non-logical parts of a theory depend on semantics (Tarski) and general philosophical views on logic and mathematics (cf. Russell)

Tarski 1941

"For precision it may be added, that the considerations which we sketched here are applicable to any deductive theory in whose construction logic is presupposed, but their application to logic itself brings about certain complications which we would rather not discuss here."

Gentzen Style vs. Hilbert Style

Reasonable systems of logic can be built with few or even no axiom(s) (= rules with the empty assumption line): Natural Deduction, Sequent Calculi

Rules are more fundamental for logic than axioms: every axiom can be seen as a rule but not the other way round.

Technical Question

Gentzen-style and Hilbert-style axiomatizations of propositional and predicate logic are equivalent (Deduction Theorem). What about a larger class of formal systems including non-logical theories axiomatized in Hilbert style? Is it always possible to reduce the number of axioms at the expense of new rules and/or vice versa?

Deduction Theorem

Deduction theorem:

$\Gamma, \phi \vdash \psi$ if and only if $\Gamma \vdash \phi \rightarrow \psi$

Constructive version (applies in MLTT):

$\Gamma, a : A \vdash b : B$ if and only if $\Gamma \vdash f : A \rightarrow B$

in words: (In the context Γ) it is possible to produce item b from given item a if and only if it is also possible to produce method f of transforming a into b.

How Deduction Theorem Fails

The above constructive principle is quite strong and doesn't apply universally. For example Euclid provides rules for constructing the regular triangle by its side but the *method* of such construction does not belong to the given theoretical domain along with straight lines, triangles and other geometrical objects. It can be studied within a metatheory but not in the geometrical theory itself.

Consider propositional logic with two variables A, B. Add to *modus ponens* new rule $A \vdash B$. This doesn't change the class of derivable formulas (since A is not a tautology and hence the new rule is never used). Then add to usual Hilbert-style axioms new axiom A. The obtained calculus derives B but not the implication $A \rightarrow B$.

OBSERVATION: The mainstream axiomatic approach outside the pure logic is Hilbert-style (since Hilbert's own works in axiomatization of Euclidean geometry)

QUESTION: Can Gentzen-style rule-based axiomatization be used for non-logical theories? (I have in mind using non-logical *rules* instead of non-logical axioms - or probably disregard the distinction between logical and non-logical elements in the axiomatic architecture altogether and leave this distinction rather to further philosophical considerations.)

Some motivations for the answer in positive:

- Controversial results of the century-long attempts to provide workable axiomatizations of physical and other scientific theories
- Euclid's theory of geometry
- The *Non-Statement View of Theories* (Suppes et al.) and the role of the *Experimental Design* (van Fraassen)
- Univalent Foundations of Mathematics / Homotopy type theory

UF/HoTT

- Simplicial UF (Voevodsky) uses rules of MLTT ("instead of axioms") plus the Axiom of Univalence (AU); in Qubical UF is (valid but) replaced by additional rules (Coquand et al. 2016). This is described by the authors as a "constructive justification" of UA.
- UF/ HoTT does not separate the MLTT syntax into the logical and the non-logical part. The original intended interpretation of this syntax is (broadly) logical. However HoTT involves a non-logical, interpretation of the same syntax, in simplicial sets.

UF/HoTT (continued)

More specifically UF supports the view according to which logic, in a narrow sense, occupies precisely at the *h*-level (-1) of the homotopical hierarchy of types: below sets, groupoids, 2-groupoids, etc.
In particular, UA at the propositional level applies as an extensionality principle known (long before the UF) as Church extensionality:

$$(p = q) \leftrightarrow (p \leftrightarrow q)$$

(in words: "equivalent propositions are equal")

UF/HoTT (continued)

(-1) truncated Π- and Σ-types are, correspondingly, universal and existential propositions as follows:

$$\|\Pi_{(x:X)}, P(x)\| \rightleftharpoons \forall x P(x); x \in X$$

$$\|\Sigma_{(x:X)}, P(x)\| \rightleftharpoons \exists x P(x); x \in X$$

Non-truncated Π- and Σ-types provide one with more information about proofs of the corresponding first-order propositions.

Ian Müller on Euclid

I know of no logic which accounts for this inference in its Euclidean formulation. One 'postulates' that a certain action is permissible and 'infers' the doing of it, he., does it. An obvious analogue of the procedure here is provided by the relation between rules of inference and a deduction. Rules of inference permit certain moves described in a general way, e.g., the inferring of a formula of the form $A \vee B$ from a formula of the form A. And in a deduction one may in fact carry out such a move, e.g., write '$(P\&Q) \vee R$' after writing '$P\&Q$'. The carrying out of a deductive step on the basis of a rule of inference is certainly not itself an inference. For neither the rule nor the step is a statement capable of truth and falsehood. And if the analogy is correct, Euclid's constructions are not inferences from his constructional postulates ; they are actions done in accord with them.

Common Notions

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

A1-A4 are to be read as rules rather than conditional axioms at the pain of Carroll Paradox of logical grounding. (*What the Tortoise Said to Achilles?* Mind, 1895)

$\frac{A=C;B=C}{A=B}$ rather then

$$\frac{[((A=C)\&(B=C))\rightarrow(A=B)];A=C;B=C}{A=B}$$

Postulates 1-3:

P1: to draw a straight-line from any point to any point.
P2: to produce a finite straight-line continuously in a straight-line.
P3: to draw a circle with any center and radius.

Postulate 1 in MLTT

Formation Rule for Pairs of Distinct Points: $\frac{Point:TYPE}{PDP:TYPE}$

Constructor for PDPs: $\frac{A,B:Point}{\langle A,B\rangle:PDP}$

Formation Rule for Lines: $\frac{\langle A,B\rangle:PDP}{Line(\langle A,B\rangle):TYPE}$
(the dependent type of lines with endpoints A, B, which depends on the PDP type)

Constructor for Lines: $\frac{\langle A,B\rangle:PDP}{|AB|:Line(\langle A,B\rangle)}$

Postulate 1 Hilbert-style

For all pairs of distinct points $\langle A, B \rangle$ there exist line $|AB|$ with endpoints A, B or in symbols (ignoring the formal typing)

$$\forall \langle A, B \rangle \exists L(ENDS(\langle A, B \rangle, L))$$

where $ENDS$ is the endpoint relation.

Back translation into MLTT:

(without trying to provide a faithful reconstruction of the binary relation END: I use instead the dependent type Line as above)

$$w : \Pi_{\langle A,B\rangle : PDP} \Sigma_{L:Line} ENDS(\langle A, B\rangle, L)$$

where $ENDS : Line \rightarrow PDP \rightarrow TYPE$ and w is an universal method (algorithm) for producing lines from from PDPs.

In this form this axiom (= rule with empty assumption) is stronger than the above system of formation rules and constructors in the sense that the latter does NOT involve *Pi*- and Σ-types. No "universal solution" such as w is needed for Euclid's theory. It is an overkill.

The "propositional translation" of Euclid's geometrical postulates is not innocent!

Ryle 1945

"[I]ntelligent reasoner is knowing rules of inference whenever he reasons intelligently [. . .] [K]nowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. Knowing a rule is knowing how." (underlining mine - A.R.)

"Anti-Intellectualism"

"Principles of inference are not extra premisses and knowing these principles exhibits itself not in the recitation of formulas but in the execution of valid inferences and in the avoidance, detection and correction of fallacies, etc. "

A good experimentalist exercises his skill not in reciting maxims of technology but in making experiment.

Gellner

“ [T]here is a tendency . . . to make knowing how do what “intuitions” used to do.”

Toulmin on tacit knowledge; the example of riding a bicycle, etc.

Stanley and Williamson 2001

"We believe that any successful account of natural language must postulate entities such as *ways*. But we shall not have much more of substance to say about the metaphysics of ways in this paper. "

The anti-intellectualist bent of this continuing debate is hardly justified. The knowledge of (i.e. how to follow) rules of logic is not "tacit" (assuming that the rules are explicitly stated). Rules do allow for linguistic representation along with propositions. Ex. Euclid's Common Notions (Axioms) and Postulates; logical rules. Linguistic examples from the everyday talk and their semantic analysis hardly explains the mutual roles of K-that and K-how.

How K-that and K-how are related?

The UF suggests the following answer: in all cases of interest the K-how is the knowledge of *justification* (= how to justify) a proposition. Only combined with a justification (which may involve K-how) a proposition can be possibly known.

Indeed, HoTT allows one to see a type of any h-level $>$ -1 as a proposition with a justification structure (= set of its proofs) and higher-order justification structure (groupoid of identities between the proofs, and so on).

Consider judgement of the form $A : Point$. It says that the type $Point$ is not empty: point A evidences this fact . Let us assume that type $Point$ is of h-level 0, i.e., is a set. Its (-1)-truncation $\|Point\|$ must be a proposition. What this proposition says (informally, in English)? It says that type $Point$ is non-empty; in other words it says that there exists a point. However $\|Point\|$ does not involve a Σ-type, which, recall, at the propositional level applies as the existential quantifier $\exists$ (or, more precisely, as a constructive version of such quantifier).

What is going on here? The (-1)-truncation $\|Point\|$ amounts to the fact that one does not distinguish between different points but only proves that such (a) thing(s) exist(s) by presenting an instance, namely point A. Point A proves proposition $\|Point\|$, which says that point(s) exist(s)!

Solution

Aristotle is right that "being is said in many ways" (πολλαχως λέγεται τὸ ὄν) and Quine is only partly right that "to be is to be a value of bound variable"

Quine is wrong that be a value of bound variable is the *only* sense existence, which is logically sound. Moreover, $\|X\|$ is clearly a more fundamental form of existential proposition than $\|\Sigma_{(x:X)}, P(x)\|$ since the latter involves the former. Primitive propositions of this form can be known by *acquaintance* (Russell).

In the standard Hilbert-style axiomatic setting this type of existence is delegated to semantics (the existence of models) and not reflected syntactically.

Conclusions

- Gentzen-style rule-based approach to axiomatization of theories provides a novel and more powerful way of formalizing theoretical ontologies;
- The epistemic role of rules in the axiomatic architecture of theories amounts to justification (grounding) of theoretical truths. Such rules, generally, do not reduce to logical rules.

Conclusions (continued)

- Since K-that is understood as *justified true (propositional) belief*, K-that depends of K-how, which provides the wanted justification. Here K-how includes the logical K-How and the knowledge of other justificatory procedures such as making observations and conducting experiments. Thus K-how plays a fundamental epistemic role in science even if one assumes that obtaining K-that is the ultimate epistemic goal. The anti-intellectual bent in the current discussion on the K-how is not justified.
- Instances of K-how can be independent of any K-that (the case of "mute" or implicit knowledge like the knowledge how to ride a bicycle. However such an autonomous K-how hardly has an epistemic significance, at least in science.

THANK YOU!