Iteration in Residuated Structures

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References: Pratt 1990, Kozen 1994.

Standard Examples

- the algebra of languages over an alphabet, possibly with the empty word;
- the algebra of binary relations, $R \subseteq W \times W$ ($\cdot$ is composition, * is the reflexive-transitive closure).

ACT$_\omega$

An infinitary sequent calculus for *-continuous action lattices.

$$A \to A \qquad \to \mathbf{1} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C}$$

$$\frac{A\,\Pi \to B}{\Pi \to A \setminus B} \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Pi\, (A \setminus B)\, \Delta \to C} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}$$

$$\frac{\Pi\, A \to B}{\Pi \to B \,/\, A} \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B \,/\, A)\, \Pi\, \Delta \to C} \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}$$

$$\frac{\Gamma \to A_1 \quad \Gamma \to A_2}{\Gamma \to A_1 \wedge A_2} \qquad \frac{\Gamma, A_i, \Delta \to C}{\Gamma, A_1 \wedge A_2, \Delta \to C}$$

$$\frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2} \qquad \frac{\Gamma, A_1, \Delta \to C \quad \Gamma, A_2, \Delta \to C}{\Gamma, A_1 \vee A_2, \Delta \to C}$$

$$\frac{\Gamma_1 \to A \quad \ldots \quad \Gamma_n \to A}{\Gamma_1, \ldots, \Gamma_n \to A^*}\ (n \geq 0) \qquad \frac{\Gamma, A^n, \Delta \to C \text{ for all } n \geq 0}{\Gamma, A^*, \Delta \to C}$$

$$\frac{\Pi \to A \quad \Gamma\, A\, \Delta \to C}{\Gamma\, \Pi\, \Delta \to C}\ (\mathrm{cut})$$

Reference: Buszkowski & Palka 2005–08

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A variant of relational semantics also exists.

Multiplicative-Only Fragment (the Lambek Calculus with Iteration, $\mathbf{L}_\omega^+$)

(for the $*$-continuous case;
cf. $\mathbf{ACT}_\omega$ by Buszkowski and Palka 2005–08)

$$A \to A$$

$$\frac{A\,\Pi \to B}{\Pi \to A \setminus B}\text{, where }\Pi\text{ is not empty} \qquad \frac{\Pi \to A \quad \Gamma\,B\,\Delta \to C}{\Gamma\,\Pi\,(A \setminus B)\,\Delta \to C}$$

$$\frac{\Pi\,A \to B}{\Pi \to B \,/\, A}\text{, where }\Pi\text{ is not empty} \qquad \frac{\Pi \to A \quad \Gamma\,B\,\Delta \to C}{\Gamma\,(B \,/\, A)\,\Pi\,\Delta \to C}$$

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CFG $\to$ Lambek categorial grammar.

$a \triangleright A$, $b_2 \triangleright B$, C is the goal category. (Alphabet $\{a, b\}$)

$a_1 \ldots a_n \in \mathcal{L} \iff A_1 \ldots A_n \to C$ is derivable.

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Open question: Safiullin's result is not known for the case with empty word. Therefore, we cannot yet replace $^+$ with *.

On The Other Side...

Pratt's axiomatisation for general (non necessarily $*$-continuous) action lattices (a variant with positive iteration):

$$A \to A \qquad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$\frac{A \to C \,/\, B}{A \cdot B \to C} \qquad \frac{A \cdot B \to C}{A \to C \,/\, B} \qquad \frac{B \to A \setminus C}{A \cdot B \to C} \qquad \frac{A \cdot B \to C}{B \to A \setminus C}$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A \to B_i}{A \to B_1 \vee B_2} \qquad \frac{A_1 \to B \quad A_2 \to B}{A_1 \vee A_2 \to B}$$

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NB: Pratt 1990 doesn't cite Lambek 1958 (but cites Girard 1987).

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Open question 2: lower complexity bounds for $\mathbf{ACT}_{\mathrm{Pratt}}$ without $\vee$ and $\wedge$.

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- $\bot^* = \bot$, $(0, 0)^* = (0, 0)$, for others $x^* = \top$.
- There is no $\sup_n (0, 1)^n$, but $(0, 1)^* = \top$ is defined.
- Any action algebra yields a positive iteration algebra with $a^+ = a \cdot a^*$.

System with Non-well-founded Derivations ($\mathbf{L}^+_\infty$)

$$\frac{\Pi \to A}{\Pi \to A^+} \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to A^+}{\Pi_1, \Pi_2 \to A^+} \qquad \frac{\Gamma, A, \Delta \to C \quad \Gamma, A, A^+, \Delta \to C}{\Gamma, A^+, \Delta \to C}$$

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- Correctness condition: for every infinite branch there exists an active occurrence A^+ that undergoes the left rule infinitely many times.

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- $\mathbf{L}^+_\infty$ is equivalent to $\mathbf{L}^+_\omega$.
- **Work in progress:** cut elimination in $\mathbf{L}^+_\infty$, cf. works of Shamkanov and Savateev.

System with Circular Proofs, $\mathbf{L}^{+}_{\mathrm{circ}}$

$$\frac{\Pi \to A}{\Pi \to A^{+}} \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to A^{+}}{\Pi_1, \Pi_2 \to A^{+}} \qquad \frac{\Gamma, A, \Delta \to C \quad \Gamma, A, A^{+}, \Delta \to C}{\Gamma, A^{+}, \Delta \to C}$$

We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (*backlink*).

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Example:

$$\dfrac{\dfrac{p, p \setminus p \to p \qquad \dfrac{p \to p \quad p, (p \setminus p)^+ \to p}{p, p \setminus p, (p \setminus p)^+ \to p}}{p, (p \setminus p)^+ \to p}}{(p \setminus p)^+ \to p \setminus p}$$

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We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (*backlink*).

Correctness condition: in each backlink, the active occurrence A^{+} in the premise should be tracked down to **the same** active occurrence in the goal.

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Open question: a cut-free system? (cf. Jipsen 2004 for a different approach).

Language Interpretation

$$w(A) \subseteq \Sigma^+$$

$$w(A \setminus B) = w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\ vu \in w(B)\}$$
$$w(B \,/\, A) = w(B) \,/\, w(A) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\ uv \in w(B)\}$$
$$w(A \cdot B) = w(A) \cdot w(B) = \{uv \mid u \in w(A), v \in w(B)\}$$
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A partial result [Ryzhkova 2013]: completeness for the fragment without $\cdot$, where $^+$ appears only as $A^+ \setminus B$ or $B \,/\, A^+$.

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- the same for relational semantics ($\cdot$ is composition, R^+ is transitive closure).

Even More Complexity: Iteration and the Exponential

The exponential, !, governed by the following rules:

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allows encoding *derivation from a finite theory* as a derivation of one formula:

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Therefore,

- **L** with ! is undecidable (Σ^0_1-complete): encoding derivations in semi-Thue systems.
- **L** with ! and * is Π^1_1-hard: encoding Kozen 2002 (deriving Horn clauses in *-continuous Kleene algebra is Π^1_1-complete).

Open question: Π^1_1 upper bound.

Thanks^+