Intuitionistic Epistemic Logic

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Objectives

Outline an intuitionistic view of knowledge which is:

1) faithful to the Brouwer-Heyting-Kolmogorov (BHK) semantics - the intrinsic semantics for intuitionstic logic, and
2) which regards **knowledge as the product of verification**.

Basic Assumptions

Intuitionistically a proposition is true if proved (BHK).

Intuitionistic knowledge is the result of verification by trusted means, which *does not necessarily produce an explicit proof of what is verified*.

Classical vs. Intuitionistic Universe

Since the classical truth of a proposition is necessary for knowledge, we have the following picture in the classical universe:

$$\textit{Classical Knowledge} \Rightarrow \textit{Classical Truth}.$$

Whereas intuitionistically we have:

$$\textit{Intuitionistic Truth} \Rightarrow \textit{Intuitionistic Knowledge}$$

The Brouwer-Heyting-Kolmogorov Semantics

A proposition, A, is true if there is a proof of it, and false if we can show that the assumption that there is a proof of A leads to a contradiction. Truth for the logical connectives is defined by the following clauses:

- a proof of $A \wedge B$ consists in a proof of A and a proof of B
- a proof of $A \vee B$ consists in giving either a proof of A or a proof B
- a proof of $A \rightarrow B$ consists in a construction which given a proof of A returns a proof of B
- $\neg A$ is an abbreviation for $A \rightarrow \bot$, and $\bot$ is a proposition that has no proof.

Incorporating constructive knowledge

If we add an epistemic (knowledge) operator $\mathbf{K}$ to our language, what should be the intended semantics of a proposition of the form $\mathbf{K}A$?

We adopt the view (cf. Williamson [9]) that

intuitionistic knowledge is the result of verification.

A verification is evidence considered sufficiently conclusive to warrant a claim to knowledge.

Intuitionistic knowledge as verification

We propose the following BHK clause for knowledge:

- a proof of $\mathbf{K}A$ is a proof of a verification that A has a proof.

  $\mathbf{K}A$, i.e. a verification of A, contains enough information to conclude that there exists a proof of A, but does not necessarily deliver that proof.

Example: inhabited types

In Intuitionistic Type Theory propositions are types whose elements are proofs (witnesses).

For each proposition type A one can form a 'truncated type' $inh(A)$ which contains no information beyond the fact that the type A is inhabited.

$\mathbf{K}A$ can be interpreted as $inh(A)$ – $\mathbf{K}A$ conveys the information that A has a proof, without delivering that proof.

Awareness issue

Traditional intuitionism assumes that proofs are available to the agent. Heyting [5] says: *"In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed''*.

Prawitz and Martin-Löf, on the other hand, assume that proofs are platonic timeless entities, truth is the existence of a proof.

If BHK proofs are assumed to be available to the agent, then $\mathbf{K}A$ can be read as *"A is known"*.

If proofs are platonic entities, not necessarily available to the knower, then $\mathbf{K}A$ is read as *"A can be known under appropriate conditions"*.

Constructive truth yields knowledge

From the BHK view of truth and implication it follows that the intuitionist should endorse the *constructivity of truth*,

$$A \to \mathbf{K}A. \qquad \text{(Co-Reflection)}$$

Why? Because **proofs are a special and most strict kind of verification**.

According to the BHK reading, $A \to \mathbf{K}A$ states that

given a proof of A one can construct a proof of $\mathbf{K}A$.

Can one always do this? Yes, because proofs are checkable.

Having checked a proof we have a proof that the proposition is proved, hence verified, i.e. known.

Verification does not yield proof

Since verification does not necessarily yield proofs

$$\mathbf{K}A \to A \qquad \text{(Reflection)}$$

is not valid as a general principle of intuitionistic epistemic logic.

Reflection states that "given a proof of $\mathbf{K}A$ one can always construct a proof of A."

Since we allow that $\mathbf{K}A$ does not necessarily produce specific proofs there is no uniform procedure which can take any adequate, non-proof, verification of A and return a proof of A.

Example: Zero-knowledge protocols

A class of cryptographic protocols, normally probabilistic, by which the prover can convince the verifier that a given statement is true, without conveying any additional information apart from the fact that that statement is true.

Example: Testimony of an authority

Take Fermat's Last Theorem. For the educated mathematician it can be claimed as known, but most mathematicians could not produce a proof of it.

More generally, it is legitimate to claim to know a theorem when one understands its content, can use it in one's reasoning, and trust that it has been verified by other mathematicians, without being in a position to produce or recite the proof.

Example: Existential generalization

Somebody stole your wallet in the subway.You have all the evidence for this: the wallet is gone, your backpack has a cut at the corresponding pocket, but you have no idea who did it.

You definitely know $\exists x S(x)$, where $S(x)$ stands for “x stole my wallet”, so $\mathbf{K}(\exists x S(x))$ holds. If intuitionistic knowledge would yield proof, you would have a constructive proof q of $\exists x S(x)$.

However, a constructive proof of the existential sentence $\exists x S(x)$ requires a witness a for x and a proof b that $S(a)$ holds. You are nowhere near meeting this requirement.

Reflection is just too strong as a truth condition

Nevertheless reflection is often taken to be practically definitive of knowledge, especially from a constructive standpoint.

- Williamson and Proietti both construct system of intuitionistic epistemic logic which affirm $\mathbf{K}A \rightarrow A$.
- Prominent philosophical anti-realists/verificationists like Wright insist that a theory of knowledge which does not validate reflection is not really about knowledge.

An obvious intention was that reflection expressed the idea that only true propositions can be known and that **false propositions cannot be known.**

False propositions cannot be known

The truth condition for knowledge can be alternatively expressed in other ways:

1. $\neg(\mathbf{K}A \wedge \neg A)$
2. $\neg A \rightarrow \neg \mathbf{K}A$
3. $\mathbf{K}A \rightarrow \neg\neg A$
4. $\neg\neg(\mathbf{K}A \rightarrow A)$
5. $\neg \mathbf{K}\bot$.

1 – 4 are classically equivalent to reflection $= \mathbf{K}A \rightarrow A$, but **intuitionistically all 1 – 5 are strictly weaker than $\mathbf{K}A \rightarrow A$.**

Correct expression of the truth condition

Intuitionistic reflection

$$\mathbf{K}A \to \neg\neg A$$

is the clearest expression of the intuitionistic truth condition on knowledge

if A is known then it is impossible that A is false.

Intuitionistic reflection is classically equivalent to reflection and hence is acceptable both classically and intuitionistically.

The double negation translation of classical logic into intuitionistic logic (cf. [2, 3, 4, 6]) and Glivenko's Theorem,

$$CPC \vdash A \Leftrightarrow IPC \vdash \neg\neg A$$

suggests the informal intuitionistic reading of $\neg\neg A$ as 'A is classically true'.

Intuitionistic reflection can be understood as claiming just what classical reflection does, i.e. that knowledge yields classical truth, *truth which does not have a specific witness*.

Accordingly **intuitionistic reflection expresses as much as its classical counterpart does**.

Intuitionistic knowledge of A is positioned strictly in between A and $\neg\neg A$:

$$A \to \mathbf{K}A \to \neg\neg A$$

If we assume that the double negation translation is a meaningful intuitionistic representation of classical truth, these findings can be presented as

$$\textit{Intuitionistic Truth} \quad \Rightarrow \quad \textit{Intuitionistic Knowledge} \quad \Rightarrow \quad \textit{Classical Truth}.$$

Though classical reflection does not hold intuitionistically nothing is lost, the intuitions that support classical reflection can be captured in an intuitionistic setting.

Intuitionistic Epistemic Logic IEL

Given the above discussion we define a system of intuitionistic epistemic logic, IEL, incorporating a BHK version of knowledge.

The language is that of intuitionistic propositional logic augmented with the propositional operator **K**.

Axioms

1. *Axioms of propositional intuitionistic logic*
2. $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$
3. $A \to \mathbf{K}A$
4. $\mathbf{K}A \to \neg\neg A$

Rules

- *Modus Ponens*

Theorem

In IEL,

1. *The rule of* $\mathbf{K}$*-necessitation,* $\vdash A \Rightarrow\ \vdash \mathbf{K}A$*, is derivable.*
2. *The Deduction Theorem holds.*
3. *Uniform Substitution holds.*
4. *Positive and Negative Introspection are valid;* $\vdash \mathbf{K}P \to \mathbf{K}\mathbf{K}P$, $\vdash \neg\mathbf{K}P \to \mathbf{K}\neg\mathbf{K}P$.

IEL^-

Let IEL^- be IEL without axiom $\mathbf{K}A \to \neg\neg A$. IEL^- is the basic intuitionistic logic of belief.

Theorem

Each of
$\mathsf{IEL}^- + \neg(\mathbf{K}A \wedge \neg A)$,
$\mathsf{IEL}^- + \neg\mathbf{K}\bot$,
$\mathsf{IEL}^- + \neg\neg(\mathbf{K}A \to A)$, *and*
$\mathsf{IEL}^- + \neg A \to \neg\mathbf{K}A$
proves $\mathbf{K}A \to \neg\neg A$, *hence each is equivalent to* IEL.

Embedding Classical Epistemic Logic into IEL

Given IEL $\vdash \neg\neg(\mathbf{K}A \to A)$ and Glivenko's Theorem it follows that the classical logic of knowledge S5 as well as logics of belief K, D, KD4, KD45 can be embedded into IEL.

The double negation of each theorem of these logics is derivable in IEL. This embedding is not faithful; IEL $\vdash \neg\neg(A \to \mathbf{K}A)$ but in none of the classical logics $\vdash A \to \mathbf{K}A$.

IEL offers a more general framework than the classical epistemic one; classical epistemic reasoning is sound in IEL, but the intuitionistic epistemic language is rather more expressive.

Limitation

In the intuitionistic propositional setting, knowledge and provably consistent belief are axiomatized by the same logical system, IEL (e.g. $\mathsf{IEL}^- + \neg \mathbf{K}\bot$).

Does this mean that intuitionistic knowledge is just provably consistent belief? Not necessarily. However, it does mean that the basic intuitionistic epistemic logic IEL does not distinguish intuitionistic knowledge from intuitionistic provably consistent belief, just like the classical epistemic logic S5 does not distinguish knowledge from true belief.

Models for IEL

Definition

A model for IEL is a quadruple $< W, R, E, \Vdash >$ such that:

1. W is a non-empty set of states.
2. R is a transitive and reflexive binary relation on W.
3. E is a binary relation s.t.
   - 3.1 $E(u)$ is non-empty;[a]
   - 3.2 $E \subseteq R$, i.e., $E(u) \subseteq R(u)$ for any state u;
   - 3.3 $R \circ E \subseteq E$, i.e., uRv yields $E(v) \subseteq E(u)$.
4. $\Vdash$ is an evaluation function such that
   - $u \nVdash \bot$;
   - for atomic p if $u \Vdash p$ and uRv then $v \Vdash p$;
   - $u \Vdash \mathbf{K}A$ iff $v \Vdash A$ for all $v \in E(u)$.

[a] Let $R(u)$ and $E(u)$ denote the R-successors and the E-successors, respectively, of some state u.

Lemma (Monotonicity)

For each formula A, if $u \Vdash A$ and uRv then $v \Vdash A$.

Proof.

Monotonicity holds for atoms and the propositional connectives, we show this just for $\mathbf{K}$.

Assume $u \Vdash \mathbf{K}p$, then $x \Vdash p$ for each $x \in E(u)$. Take an arbitrary v such that uRv and arbitrary w such that vEw. By the condition on $R \circ E$ in the model, $uRvEw$ yields uEw, hence $w \in E(u)$. Therefore, $w \Vdash p$ and hence $v \Vdash \mathbf{K}p$. ∎

Theorem (Soundness and Completeness)

$\mathsf{IEL} \vdash A \Leftrightarrow \mathsf{IEL} \Vdash A.$

Induction on derivations and via the cannonical model construction, respectively.

Theorem

IEL $\nvdash \mathbf{K}p \to p$.

Proof.

Consider the following model:
$1R2$, R is reflexive and transitive,
$E(1) = E(2) = \{2\}$,
p is atomic and $2 \Vdash p$.

Clearly, $1 \Vdash \mathbf{K}p$ and $1 \nVdash p$.

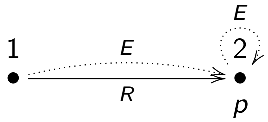


Figure: Model $\mathcal{M}_1$

Theorem (Reflection for negative formulas)

$\mathsf{IEL} \vdash \mathbf{K}\neg A \to \neg A.$

Theorem

The rule $\vdash \mathbf{K}A \;\Rightarrow\; \vdash A$ *is admissible.*

In IEL knowledge and negation commute. The impossibility of verifying A is equivalent to verifying that A cannot possibly hold.

Theorem

IEL $\vdash \neg \mathbf{K}A \leftrightarrow \mathbf{K}\neg A$.

In IEL the impossibility of verification is equivalent to the impossibility of proof.

Theorem

IEL $\vdash \neg \mathbf{K}A \leftrightarrow \neg A$.

In IEL no truth is unverifiable (all truths are knowable).

Theorem

IEL $\vdash \neg(\neg \mathbf{K}A \wedge \neg \mathbf{K}\neg A)$.

Proof.

1. $\neg \mathbf{K}A \wedge \neg \mathbf{K}\neg A$ - assumption;
2. $\mathbf{K}\neg A \wedge \mathbf{K}\neg\neg A$ - by Theorem 9;
3. $\neg A \wedge \neg\neg A$ - by Theorem 7;
4. $\bot$ - 3;
5. $\neg(\neg \mathbf{K}A \wedge \neg \mathbf{K}\neg A)$ - 1–4. ∎

Intuitionistic verifications do not have the disjunction property.

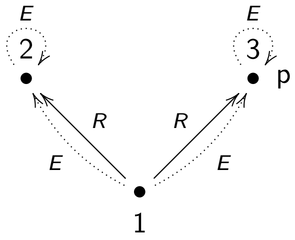
Theorem

IEL $\nvdash \mathbf{K}(A \vee B) \rightarrow (\mathbf{K}A \vee \mathbf{K}B)$.

Proof.

Consider the following model.
$1R2, 1R3$;
$1E2, 1E3, 2E2, 3E3$;
p is atomic and $3 \Vdash p$.

Since $2 \nVdash p$, $2 \Vdash \neg p$, hence $2 \Vdash p \vee \neg p$. Since $3 \Vdash p$ $3 \Vdash p \vee \neg p$. Hence $1 \Vdash \mathbf{K}(p \vee \neg p)$. However, $1 \nVdash \mathbf{K}p$, and $1 \nVdash \mathbf{K}\neg p$.



Figure

∎

Theorem (Disjunction Property)

If IEL $\vdash A \vee B$ *then either* IEL $\vdash A$ *or* IEL $\vdash B$.

Despite Theorem 12, IEL has a weak disjunction property for verifications.

Corollary

If $\vdash \mathbf{K}(A \vee B)$ *then either* $\vdash \mathbf{K}A$ *or* $\vdash \mathbf{K}B$.

Proof.

Assume IEL $\vdash \mathbf{K}(A \vee B)$ then, by the reflection rule, $\vdash A \vee B$, hence $\vdash A$ or $\vdash B$. In which case $\vdash \mathbf{K}A$ or $\vdash \mathbf{K}B$ by **K**-necessitation. ∎

Knowability

A generalisation of the constructive idea of truth is to say that the truth of a proposition is given the conditions under which it is verified.

If the truth of a proposition is determined by its verification conditions, then for it to be true it must be possible to know it.

The characteristic slogan about constructive truth is:

"All truths are knowable"

$$F \rightarrow \Diamond \mathbf{K} F.$$

The "knowability paradox" is a proof, due to Church and Fitch, which purports to show that *"all truths are knowable"* implies *"all truths are known."*

I.e. that the *Principle of Verificationist Knowability*:

$$F \to \Diamond \mathbf{K} F \qquad (VK)$$

yields

Omniscience Principle:

$$F \to \mathbf{K} F \qquad (OMN)$$

Theorem (Church-Fitch)

VK as schema yields OMN.

Proof.

1. $(p \wedge \neg \mathbf{K}p) \rightarrow \Diamond \mathbf{K}(p \wedge \neg \mathbf{K}p)$ - verificationist knowability;
2. $\mathbf{K}(p \wedge \neg \mathbf{K}p)$ - assumption;
3. $\mathbf{K}p \wedge \mathbf{K}\neg \mathbf{K}p$ - from 2 by standard modal reasoning;
4. $\mathbf{K}p \wedge \neg \mathbf{K}p$ - from 3 and reflection;
5. $\neg \mathbf{K}(p \wedge \neg \mathbf{K}p)$ - from 2–4;
6. $\Box \neg \mathbf{K}(p \wedge \neg \mathbf{K}p)$ - from 5 and necessitation;
7. $\neg \Diamond \mathbf{K}(p \wedge \neg \mathbf{K}p)$ - from 6 and $\Box \neg X \rightarrow \neg \Diamond X$;
8. $\neg(p \wedge \neg \mathbf{K}p)$ - from 1 and 7;
9. $p \rightarrow \neg\neg \mathbf{K}p$ - from 8 and $\neg(X \wedge Y) \rightarrow (X \rightarrow \neg Y)$;
10. $p \rightarrow \mathbf{K}p$ - from 9 and double negation elimination.

∎

Some consider the principle *"all truths are knowable"* as a characterization of a constructive view of truth.

If it really implies *"all truths are known"* then this view of truth does not appear to be correct.

Intuitionistic responses (like [7, 8, 9]) argue that intuitionists must reject $A \to \mathbf{K}A$.

The proper intuitionistic response is simply that there is no paradox; intuitionistically the ‘knowability paradox’ is a pseudo-problem which holds *only* from a classical standpoint.

The ‘knowability paradox’ depends on the following assumptions:

1. $A \rightarrow \mathbf{K}A$ means *all truths are known*.
2. $A \rightarrow \Diamond\mathbf{K}A$ means *all truths are knowable*.
3. That all truths are knowable is definitive of intuitionistic truth.

Response to 1

$A \to \mathbf{K}A$ can be understood as claiming *all truths are known* only on a classical reading. Intuitionistically it means something very different, namely, *constructive truth, i.e. proof, yields verification/knowledge*, which is central to an intuitionistic view of knowledge.

Response to 2

$A \to \Diamond \mathbf{K} A$ is not a good classical formal characterisation of intuitionistic truth. As a classical principle it does not capture the intended (intuitionistic) relation between truth and knowledge.

The straightforward classical logic reading of $A \to \Diamond \mathbf{K} A$ says *all classical truths are knowable*, which is plainly false.

Gödel's translation of IPC to S4 suggests that $\Box A \to \Diamond \mathbf{K} A$ is a more appropriate classical representation, c.f. [1].

Response to 3

The characteristic feature of intuitionistic truth is its constructivity. Arguably $A \to \mathbf{K}A$ is a formal representation of this.

Intuitionistic truth is knowable *because it is constructive*. Constructivity, not knowability, is the definitive feature of intuitionistic truth.

$\neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$ or $A \to \neg\neg\mathbf{K}A$ might be understood as saying "all truths are knowable", but each is an easy consequence of $A \to \mathbf{K}A$.

Thank you!

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