Extra-logical proof-theoretic semantics in HoTT

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Kant on Geometrical Proofs

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on Geometrical Proofs

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [..] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (KRV : A 716 / B 744)

Kant on Discursive and Constructive Reasoning

In these examples we have only attempted to make distinct what a great difference there is between the discursive use of reason in accordance with concepts and its intuitive use through the construction of concepts. Now the question naturally arises, what is the cause that makes such a twofold use of reason necessary, and by means of which conditions can one know whether only the first or also the second takes place? (KRV : A 719 / B 747)

Hilbert and Bernays on Axiomatic and Genetic Theories

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert and Bernays on Axiomatic and Genetic Theories

In Hilbert's Foundations of Geometry [of 1899] the axiomatic standpoint received a sharpening regarding the axiomatic development of a theory: From the factual and conceptual subject matter that gives rise to the basic notions of the theory, we retain only the essence that is formulated in the axioms, and ignore all other content. Finally, for axiomatics in the narrowest sense, the existential form comes in as an additional factor. This marks the difference between the axiomatic method and the constructive or genetic method of grounding a theory.

Hilbert and Bernays on Axiomatic and Genetic Theories

While the constructive method introduces the objects of a theory only as a genus of things, an axiomatic theory refers to a fixed system of things (or several such systems), and for all predicates of the propositions of the theory, this fixed system of things constitutes a delimited domain of subjects, about which hold propositions of the given theory. There is the assumption that the domain of individuals is given as a whole. Except for the trivial cases where the theory deals only with a finite and fixed set of things, this is an idealising assumption that properly augments the assumptions formulated in the axioms.

Hilbert and Bernays on Axiomatic and Genetic Theories

We will call this sharpened form of axiomatics (where the subject matter is ignored and the existential form comes in) formal axiomatics for short. (Hilbert & Bernays 1934, Intro)

Markov Jr. on Constructive Mathematics

The constructive trend in mathematics significantly developed during the last years. It's essence is that only constructive objects along with the abstraction of their potential realisation (without the abstraction of actual infinity) are considered. Purely existential theorems are rejected since the existence proofs require a specification of potentially realisable method of constructing an object with required properties. Constructive objects are figures built from elementary figures, i.e., elementary constructive objects. A simple example of constructive object is a word constructed with a fixed alphabet. A word in a given alphabet is a sequence of letters of this alphabet. (1962)

Constructive syntax or constructive semantics?

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

MLTT: Semantics of $t : T$ (Martin-Löf 1984)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{refl_x : x =_A y}$$

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher & Hofmann 1995).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\dots$

HoTT: the Idea

Types in MLTT are modelled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of *n*-groupoids as higher homotopy groupoids of an appropriate topological space).

Homotopical interpretation of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).

Homotopy Levels

Definition

We say that S is a space of h-level $n+1$ if for all its points x, y path spaces $x =_S y$ are of h-level n.

Cummulative Hierarchy of Homotopy Types

- -2-type: single point *pt*;
- -1-type: the empty space $\emptyset$ and the point *pt*: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Remark

In addition to the syntactic bottom-up intuition about higher types (cf. paper by Paolo Pistone & Luca Tranchini at this conference) HoTT provides this useful top-down geometrical intuition: n-types finitary approximate the topological structure of the base space, which by default is an ω-type. Cf. the Taylor expansion of a given differentiable function.

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ entails $x \equiv y$ (definitionally).

(Internal criterion of logicality)

Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- "Merely" logical rules (i.e., rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Qs & As

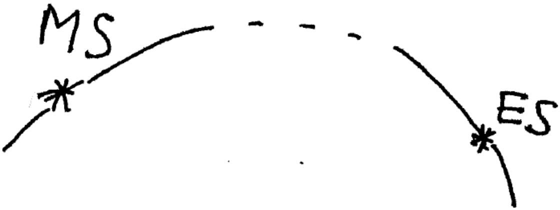
- Q: What is a possible PTS for non-logical theories? (Sara Negri, Peter Schröder-Heister)
- A: It is a geometrical (to wit, homotopical) semantics, which interprets HoTT rules for non-propositional types as rules for geometrical constructions. Cf. Euclid's rules aka postulates for constructing figures by the ruler and compass.
- Q: Does this semantics really qualify as proof-theoretic?
- A: Yes because the higher-order extra-logical constructions prove their underlying propositions (obtained via the propositional truncation).
- Q: What validates these rules in the cases of their extra-logical application?
- A: Geometrical Intuition!

Cassirer 1907 on mathematical intuition

> *The principle according to which our concepts should be sourced in intuitions means that they should be sourced in the mathematical physics and should prove effective in this field. Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science itself. (p. 43-44)*

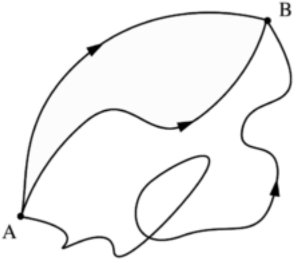
IF HoTT/UF is a satisfactory formal foundation of mathematics and IF mathematics (rather than pure logic) is a formal foundation of all science then the intuitive grounding of HoTT rules qualifies as topic-neutral also in extra-logical applications of these rules.

Back to *Principia Mathematica*?



Venus Homotopically http://philsci-archive.pitt.edu/12116/

also in the Quantum Realm?



PTS does not replace the Model theory...

(Göran Sundholm, Zhaohui Luo)

Interpretation of rules: MTS or PTS?

Standard version:
Interpretation m is a model of rule R

$$\frac{A_1^m, \ldots, A_n^m}{B^m} \tag{1}$$

when the following holds: whenever $A_1^m, \ldots, A_n^m$ are true statements B^m is also true statement. (Tsementzis 2017)

Models of HoTT according to Voevodsky

Idea: Theories are generic models (Lawvere 1963 : Functorial Semantics)

Models of HoTT according to Voevodsky

(1) Construct a general model of given type theory **T** (MLTT or its variant) as a category $\mathcal{C}$ with additional structures which model **T**-rules. For that purpose the authors use the notion of *contextual category* due to Cartmell 1978; in later works Voevodsky uses a modified version of this concept named by the author a *C-system*.

Models of HoTT according to Voevodsky

(2) Construct a particular contextual category (variant: a C-system) $\mathcal{C}(\mathbf{T})$ of syntactic character, which is called *term model*. Objects of $\mathcal{C}(\mathbf{T})$ are MLTT-contexts, i.e., expressions of form

$$[x_1 : A_1, \ldots, x_n : A_n]$$

taken up to the definitional equality and the renaming of free variables and its morphisms are substitutions (of the contexts into $\mathbf{T}$-rule schemata) also identified up to the definitional equality and the renaming of variables). More precisely, morphisms of $\mathcal{C}(T)$ are of form

Models of HoTT after Voevodsky

$$f : [x_1 : A_1, \dots, x_n : A_n] \to [y_1 : B_1, \dots, y_m : B_m]$$

where f is represented by a sequent of terms $f_1, \dots, f_m$ such that
$x_1 : A_1, \dots, x_n : A_n \vdash f_1 : B_1$
$\vdots$
$x_1 : A_1, \dots, x_n : A_n \vdash f_m : B_m(f_1, \dots, f_m)$
Thus morphisms of $\mathcal{C}(T)$ represent derivations in **T**.

Models of HoTT after Voevodsky

- Define an appropriate notion of morphism between contextual categories (C-systems) and form category $CTXT$ of such categories.
- Show that $\mathcal{C}(\mathbf{T})$ is initial in $CTXT$, that is, that for any object $\mathcal{C}$ of $CTXT$ there is precisely one morphism (functor) of form $\mathcal{C}(\mathbf{T}) \to \mathcal{C}$.

The last item is the **Initiality Conjecture** that presently stands open.

Conclusion 1

HoTT is as a "genetic" theory in Hilbert's sense. It involves an extra-logical homotopical semantics as well as a clear logical structure and logical semantics. This semantics qualifies as proof-theoretic.

Conclusion 2

The model theory of HoTT remains a work in progress. The novel notions of theory and model, which emerge in this context, require a further sharpening both mathematically and philosophically.

Thank You! Danke! Спасибо!