

Proof-Verification & Mathematical Intuition

Lecture 1. From Euclid to Hilbert

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1 August

Introduction

Euclid

Intuition under Suspicion : Lobachevsky et al.

Hilbert

Mathematical Intuition in Physics

Plan of 2 Lectures :

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1. From Euclid to Hilbert : Brief History of Axiomatic Method

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1. From Euclid to Hilbert : Brief History of Axiomatic Method
2. From Bourbaki to Voevodsky : Univalent Foundations of Mathematics.

Plan of Lecture 1

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Hilbert

Mathematical Intuition in Physics

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- ▶ Form and Content of Proofs ;

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- ▶ Proof-Verification is pleonastic! Is it a symptom or diagnosis?
- ▶ Form and Content of Proofs;
- ▶ Social and Political Relevance of PV.

Kant on Euclidean proofs

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on Euclidean proofs (cont'd)

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [...] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question.” (*Critique of Pure Reason* A 716 / B 744)

Euclid's First Principles

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► Definitions

Euclid's First Principles

- ▶ Definitions
- ▶ Postulates

Euclid's First Principles

- ▶ Definitions
- ▶ Postulates
- ▶ Common Notions (aka Axioms)

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2. to produce a finite straight-line continuously in a straight-line ;
3. to draw a circle with any centre and radius ;
4. that all right-angles are equal to one another ;
5. that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side.

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- ▶ things coinciding with one another are equal to one another ;
- ▶ the whole is greater than the part.

Proclus (412-485 A.D.) on Euclidean proofs

Every Problem and every Theorem that is furnished with all its parts should contain the following elements : an *enunciation*, an *exposition*, a *specification*, a *construction*, a *proof*, and a *conclusion*. Of these *enunciation* states what is given and what is being sought from it, a perfect *enunciation* consists of both these parts. The *exposition* takes separately what is given and prepares it in advance for use in the investigation. The *specification* takes separately the thing that is sought and makes clear precisely what it is. The *construction* adds what is lacking in the given for finding what is sought. The *proof* draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The *conclusion* reverts to the *enunciation*, confirming what has been proved. (Proclus, Comm. on Euclid)

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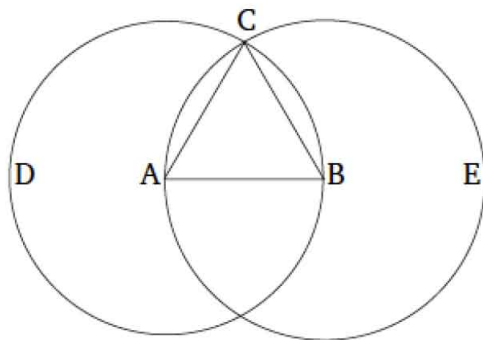
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- ▶ **specification** : So it is required to construct an equilateral triangle on the straight-line AB .
- ▶ **construction** : Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C , where the circles cut one another, to the points A and B [Post. 1].

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- ▶ **proof** : And since the point A is the center of the circle CDB , AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE , BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB . Thus, CA and CB are each equal to AB . But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB . Thus, the three (straight-lines) CA , AB , and BC are equal to one another.

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- ▶ **conclusion** : Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB . (Which is) the very thing it was required to do.

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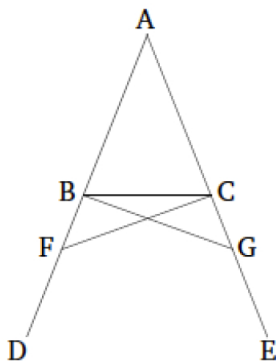
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- ▶ **construction** : For let a point F be taken somewhere on BD , and let AG have been cut off from the greater AE , equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC , GB have been joined. [Post. 1]

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► **proof :**

In fact, since AF is equal to AG , and AB to AC , the two (straight lines) FA , AC are equal to the two (straight lines) GA , AB , respectively. They also encompass a common angle FAG . Thus, the base FC is equal to the base GB , and the triangle AFC will be equal to the triangle AGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG , and AFC to AGB .

Euclid : Proposition (Theorem) 1.5

► proof (cont'd)

And since the whole of AF is equal to the whole of AG , within which AB is equal to AC , the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB . So the two (straight lines) BF , FC are equal to the two (straight lines) CG , GB respectively, and the angle BFC (is) equal to the angle CGB , while the base BC is common to them.

Thus the triangle BFC will be equal to the triangle CGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4].

Euclid : Proposition (Theorem) 1.5

► **proof (cont'd)**

Thus FBC is equal to GCB , and BCF to CBG . Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF , within which CBG is equal to BCF , the remainder ABC is thus equal to the remainder ACB [Ax. 3]. And they are at the base of triangle ABC . And FBC was also shown (to be) equal to GCB . And they are under the base.

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- ▶ **conclusion** : Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

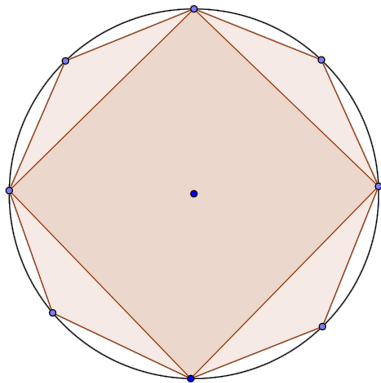
Proclus on Problems and Theorems

Some of the ancients, however, such as the followers of Speusippus and Amphinomus, insisted on calling all propositions theorems, considering theorems to be a more appropriate designation than problems for the objects of the theoretical sciences, especially since these sciences deal with eternal things. [..]. [I]t is better, according to them, to say that all these objects exist and that we look on our construction of them not as making, but as understanding them [..] Others, on the contrary, such as the mathematicians of the school of Menaechmus, thought it correct to say that all inquiries are problems but that problems are twofold in character : sometimes their aim is to provide something sought for, and at other times to see, with respect to determinate object, what or of what sort it is, or what quality it has, or what relations it bears to something else. Both parties are right.

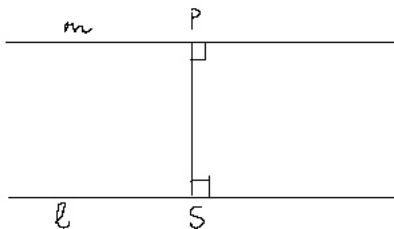
Ian Müller (1974) on logic on Euclidean proofs

I know of no logic which accounts for this inference in its Euclidean formulation. One 'postulates' that a certain action is permissible and 'infers' the doing of it, i.e., does it. (on P1)

Chiliagon and $(2^{10}=1024)$ -gon



Lobachevsky on Problem of Parallels



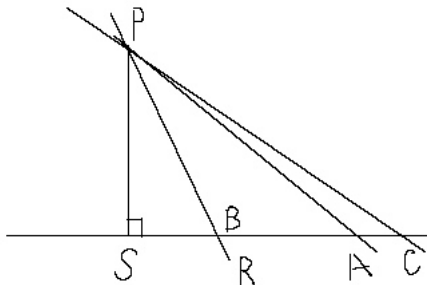
Lobachevsky on Problem of Parallels

- ▶ Given a straight line and a point outside this line there exist secants of the given line passing through the given point. To construct a secant take any point of the given line and connect it to the given point outside this line.

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- ▶ Given a straight line and a point outside this line there exist secants of the given line passing through the given point. To construct a secant take any point of the given line and connect it to the given point outside this line.
- ▶ Let PS be perpendicular to l and A be a point of l . Consider a straight line PR such that angle SPR is a proper part of angle SPA (and hence is less than angle SPA). Given this I shall call line PR *lower* than line PA (and call PA *upper* than PR). Notice that this definition involves the perpendicular PS , and so depends on the choice of P . Then PR intersects l in some point B , i.e. it is a secant. In other words a line, which is lower than a given secant is also a secant.

Lobachevsky on Problem of Parallels



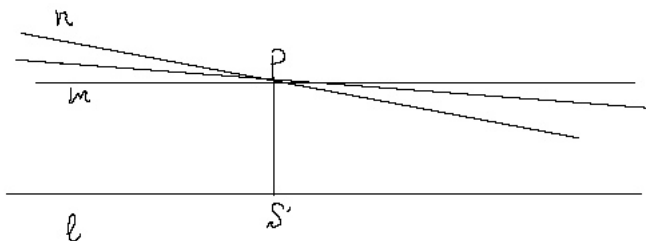
Lobachevsky on Problem of Parallels

- ▶ There exist no upper bound for secants of a given line passing through a given point outside this given line. For given some secant PA one can always take a further point C such that A will lay between S and C and so secant PC be upper than the given secant PA .

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- ▶ Let m be parallel to l , which is constructed as in (i). Let n be another parallel to l passing through the same point P . Suppose that n is lower than m (obviously this condition doesn't restrict the generality). Then any straight line which is upper than n and lower than m is also parallel to l

Lobachevsky on Problem of Parallels



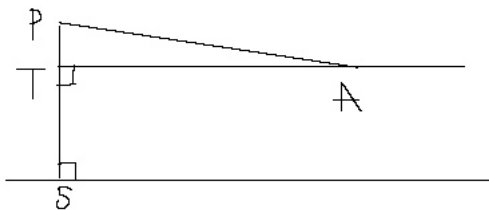
Lobachevsky on Problem of Parallels

Parallels to a given straight line passing through a given point have a lower bound. To assure it rigorously one needs some continuity principle like one asserting the existence of Dedekind cuts. Then (vi) follows from (iv). Lobachevsky doesn't states such a principle explicitly but endorses (vi) anyway.

Lobachevsky on Problem of Parallels

Any straight line PA (a secant or a parallel) passing through point P as shown at is wholly characterised by its characteristic angle SPA . Let the measure of SPA corresponding to the case of the lowest parallel be α . Now it is clear that by an appropriate choice of l and P one can make α as close to $\pi/2$ as one wishes. For given any angle $SPA < \pi/2$ it is always possible to drop perpendicular AT on PS (Fig.4). Then PA is a secant of AT and so all parallels to AT including its lowest parallels are upper than PA . Hence the value of α corresponding to straight line AT and point P outside this line is between SPA and $\pi/2$. Since the only variable parameter of the configuration is the distance d between the given straight line and the given point outside this line the angle α is wholly determined by this distance.

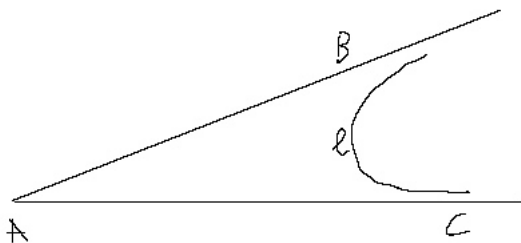
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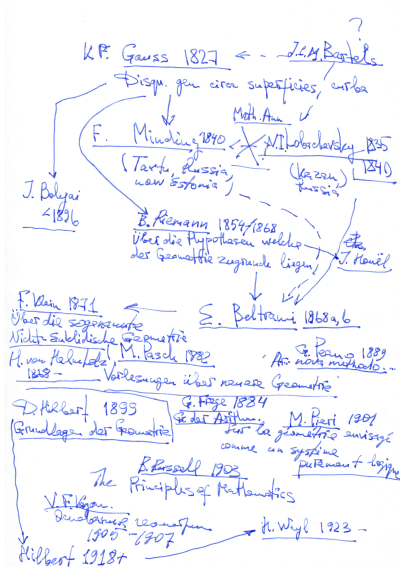


Lobachevsky on Problem of Parallels

If P5 doesn't hold then given an angle ABC , however small, there always exist a straight line l laying wholly inside this angle and intersecting none of its two sides.

Lobachevsky on Problem of Parallels





Weyl (1923) on Riemann's legacy

The question of the validity of the “fifth postulate”, on which historical development started its attack on Euclid, seems to us nowadays to be a somewhat accidental point of departure. The knowledge that was necessary to take us beyond the Euclidean view was, in our opinion, revealed by Riemann. (H. Weyl 1923)

Hilbert 1899 Foundations of Geometry : Intuition Expelled

Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space.

Hilbert 1899 Foundations of Geometry

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C, \dots ; those of the second, we will call straight lines and designate them by the letters a, b, c, \dots ; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . [...] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as “are situated”, “between”; “parallel”, “congruent”, “continuous”, etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms [...] express certain related fundamental facts of our intuition.

Hilbert 1899 Foundations of Geometry

Axiom 1.1 : Two distinct points A, B always completely determine a straight line a . We write $AB = a$ or $BA = a$.

[Hilbert's Comment] : Instead of “determine”, we may also employ other forms of expression ; for example, we may say A “lies upon” a, A “is a point of” a, a “goes through” A “and through” B, a “joins” A “and” or “with” B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression : “The straight lines” a “and” b “have the point A in common”, etc.

Hilbert 1899 Foundations of Geometry

Theorem : Two [different] straight lines have either one point or no point in common.

Proof : For let lines a, b have two different points in common. Then by Ax. 1.1 $a = b$. Hence a, b do not have two different points in common. Hence a, b have either one point in common or a, b no point in common.]

Remark : The reasoning requires the identity concept but not number concept.

Ax 1.1 formalised with CFOL with sorts

$A, B : POINT$

$a, b : LINE$

$INC[(A, a) \& INC(A, b) \& INC(B, a) \& INC(B, b)] \rightarrow (a = b)$

Non-standard interpretation : lines for “points” and points for “lines”

Next step : “axiomatization of logic” with a symbolic calculus

[I]t appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic. This method was prepared long ago (not least by Frege’s profound investigations); it has been most successfully explained by the acute mathematician and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the axiomatization of logic as the crowning achievement of the work of axiomatization as a whole. (Hilbert 1917)

Epistemic Requirement : Consistency

Proof of (relative) consistency : exhibiting a model in a theory, consistency of which is taken for granted (informal arithmetic).

Epistemic Desideratum : Independence of Axioms

Minimising the number of axioms

Next step : “axiomatization of logic” with a symbolic calculus

Motivations :

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Motivations :

- ▶ Clarification of the background logic

Next step : “axiomatization of logic” with a symbolic calculus

Motivations :

- ▶ Clarification of the background logic
- ▶ Proof of absolute consistency (before Gödel’s Second Incompleteness Theorem ; but see Artemov 2019)

Next step : “axiomatization of logic” with a symbolic calculus

Mathematical logic, also called *symbolic logic* or *logistic*, is an extension of the formal method of mathematics to the field of logic. It employs for logic a symbolic language like that which has long been in use to express mathematical relations. In mathematics it would nowadays be considered Utopian to think of using only ordinary language in constructing a mathematical discipline. The great advances in mathematics since antiquity, for instance in algebra, have been dependent to a large extent upon success in finding a usable and efficient symbolism.

The next step : “axiomatization of logic” with a symbolic calculus

The purpose of the symbolic language in mathematical logic is to achieve in logic what it has achieved in mathematics, namely, an exact scientific treatment of its subject-matter. The logical relations which hold with regard to judgments, concepts, etc., are represented by formulas whose interpretation is free from the ambiguities so common in ordinary language.

The next step : “axiomatization of logic” with a symbolic calculus

The transition from statements to their logical consequences, as occurs in the drawing of conclusions, is analyzed into its primitive elements, and appears as a formal transformation of the initial formulas in accordance with certain rules, similar to the rules of algebra ; logical thinking is reflected in a logical calculus. This calculus makes possible a successful attack on problems whose nature precludes their solution by purely contentful [*inhaltliche*] logical thinking. Among these, for instance, is the problem of characterizing those statements which can be deduced from given premises.

Next step : “axiomatization of logic” with a symbolic calculus

With this new way of providing a foundation for mathematics, which we may appropriately call a proof theory, I pursue a significant goal, for I should like to eliminate once and for all the questions regarding the foundations of mathematics in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science.

Intuition strikes back !

No more than any other science can mathematics be founded by logic alone ; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction.

Intuition strikes back !

This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed ; no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not.
(Hilbert 1927)

Hilbert-style deduction system : Rule

Rule : Modus Ponens :

$$\frac{A, A \rightarrow B}{B}$$

Hilbert-style deduction system : Axioms'' (axiom schemas)

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► P1 : $\phi \rightarrow \phi$

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ P1 : $\phi \rightarrow \phi$
- ▶ P2 : $\phi \rightarrow (\psi \rightarrow \phi)$

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ P1 : $\phi \rightarrow \phi$
- ▶ P2 : $\phi \rightarrow (\psi \rightarrow \phi)$
- ▶ P3 : $(\phi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \xi))$

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ P1 : $\phi \rightarrow \phi$
- ▶ P2 : $\phi \rightarrow (\psi \rightarrow \phi)$
- ▶ P3 : $(\phi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \xi))$
- ▶ P4 : $(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ Q5 : $\forall x (\phi) \rightarrow \phi[x := t]$ where 't' may be substituted for 'x' in ϕ

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ Q5 : $\forall x (\phi) \rightarrow \phi[x := t]$ where 't' may be substituted for 'x' in ϕ
- ▶ Q6 : $\forall x (\phi \rightarrow \psi) \rightarrow (\forall x (\phi) \rightarrow \forall x (\psi))$

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ Q5 : $\forall x (\phi) \rightarrow \phi[x := t]$ where 't' may be substituted for 'x' in ϕ
- ▶ Q6 : $\forall x (\phi \rightarrow \psi) \rightarrow (\forall x (\phi) \rightarrow \forall x (\psi))$
- ▶ Q7 : $\phi \rightarrow \forall x (\phi)$ where 'x' is not free in ϕ .

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ I8 : $x = x$ for every variable

Hilbert-style deduction system : Axioms'' (axiom schemas)

- ▶ I8 : $x = x$ for every variable
- ▶ I9 : $(x = y) \rightarrow (\phi[z := x] \rightarrow \phi[z := y])$

Example : Proof (= formal derivation) of P1

from P2, P3 : B_1, B_2, B_3, B_4, B_5

Example : Proof (= formal derivation) of P1

from P2, P3 : B_1, B_2, B_3, B_4, B_5

- ▶ $B_1 : (A \rightarrow ((A \rightarrow A) \rightarrow A)) \text{ to } ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
(P3 for $\phi := A$ and $\psi := A \rightarrow A$ and $\xi := A$)

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- ▶ $B_2 : (A \rightarrow (A \rightarrow A) \rightarrow A)$ (P2 for $\phi := A$ and $\psi := A \rightarrow A$)

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- ▶ $B_4 : (A \rightarrow (A \rightarrow A))$
- ▶ $B_5 : (A \rightarrow A)$ (MP from B_3 and B_4)

ZFC1 : Extensionality

$$\forall X \forall Y [X = Y \Leftrightarrow \forall z (z \in X \Leftrightarrow z \in Y)]$$

ZFC2 : Pairing

$$\forall x \forall y \exists Z \forall z [z \in Z \Leftrightarrow z = x \text{ or } z = y]$$

ZFC3 : Union

$$\forall X \exists Y \forall y [y \in Y \Leftrightarrow \exists Z (Z \in X \text{ and } y \in Z)]$$

ZFC4 : the Empty set

$\exists X \forall y [y \notin X]$ (this set X is denoted by \emptyset)

ZFC5 : Infinity

$$\exists X [\emptyset \in X \text{ and } \forall x(x \in X \Rightarrow x \cup \{x\} \in X)]$$

ZFC6 : Power set

$$\forall X \exists Y \forall Z [Z \in Y \Leftrightarrow \forall z (z \in Z \Rightarrow z \in X)]$$

ZFC7 :Replacement

$$\forall x \in X \exists ! y P(x, y) \quad \Rightarrow \quad [\exists Y \forall y (y \in Y \Leftrightarrow \exists x \in X (P(x, y)))]$$

ZFC8 Regularity

$$\forall X [X \neq \emptyset \Rightarrow \exists Y \in X (X \cap Y = \emptyset)]$$

ZFC9 : Axiom of choice

$$\forall X [\emptyset \notin X \text{ and } \forall Y, Z \in X (Y \neq Z \Rightarrow Y \cap Z = \emptyset) \Rightarrow \exists Y \forall Z \in X \exists! z \in Z (z \in Y)]$$

 \Rightarrow

ZFC : syntactic conventions (definitions)

\forall = for all $\exists!$ = there exists a unique P is any formula that
does not contain Y

$$z \in X \cup Y \Leftrightarrow z \in X \text{ or } z \in Y$$

$$z \in X \cap Y \Leftrightarrow z \in X \text{ and } z \in Y$$

Aristotle on proofs

By demonstration I mean a syllogism productive of scientific knowledge, a syllogism, that is, the grasp of which is eo ipso such knowledge. . . . [T]he premisses of demonstrated knowledge must be true, primary, immediate, better known than and prior to the conclusion, which is further related to them as effect to cause. Unless these conditions are satisfied, the basic truths will not be appropriate to the conclusion. Syllogism there may indeed be without these conditions, but such syllogism, not being productive of scientific knowledge, will not be demonstration. (An. Post. Book 1, ch.2, transl. G.R.G. Mure)

Prawitz (1979) on formal proofs

It is true that the question whether something is a proof must depend on the meaning of the sentences involved. And obviously, a valid argument must preserve truth. But the preservice of truth is clearly not a sufficient condition for validity; nobody would consider e.g. Peano's axioms followed by Fermat's last theorem as a proof, even if in fact Fermat's last theorem follows from these axioms. As every examiner stresses, it is not enough that the steps of a proof happen to follow from the preceding ones, it must also be seen that they follow. Nobody would consider e.g. Peano's axioms followed by Fermat's last theorem as a proof, even if in fact Fermat's last theorem follows [is formally derivable] from these axioms.

Essenin-Volpin (1970) on proofs

By proof of a judgement I mean a honest procedure making this judgement inarguable.

Geometry and Physics : Lobachevsky (NFG 1835)

[T]he assumption according to which some natural forces follow one Geometry while some other forces follow some other specific Geometry, which is their proper Geometry, cannot bring any contradiction into our mind.

Geometry and Physics : Hilbert (1894)

Among the appearances or facts of experience manifest to us in the observation of nature, there is a peculiar type, namely, those facts concerning the outer shape of things, Geometry deals with these facts [..]. Geometry is a science whose essentials are developed to such a degree, that all its facts can already be logically deduced from earlier ones. Much different is the case with the theory of electricity or with optics, in which still many new facts are being discovered. Nevertheless, with regards to its origins, geometry is a natural science (Hilbert 1894)

Geometry and Physics : Cassirer (1907)

According to Russell [1903] even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions. Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos.

Geometry and Physics : Cassirer (1907)

[According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. [...]

Geometry and Physics : Cassirer (1907)

The principle according to which our concepts should be sourced in **intuitions** means that they should be sourced in the mathematical physics and should prove effective in this field. Logical and mathematical concepts must no longer produce instruments for building a metaphysical “world of thought” : their proper function and their proper application is only within the **empirical science itself**.

Thank you !