

## Constructive Axiomatic Method in Euclid, Hilbert and Voevodsky

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The version of axiomatic method stemming from Hilbert [Hilbert (1899)] and recently defended by Hintikka [Hintikka (2011)] is not fully adequate to the recent successful practice of axiomatizing mathematical theories. In particular, the axiomatic architecture of Homotopy Type theory (HoTT) [Voevodsky et al. 2013] does not quite fit the standard Hilbertian pattern of formal axiomatic theory. At the same time HoTT and some other recent theories fall under a more general and in some respects more traditional notion of axiomatic theory, which I call after Hilbert and Bernays [Hilbert&Bernays (1934-1939)] “genetic” or “constructive” (interchangeably) and demonstrate it using the classical example of the First Book of Euclid’s “Elements”. On the basis of these modern and ancient examples I claim that Hintikka’s semantic-oriented formal axiomatic method is not self-sustained but requires a support of some more basic constructive method. I provide an independent epistemological grounding for this claim by showing the need to complement Hintikka’s account of axiomatic method with a constructive notion of formal semantics.

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## Geometric reasoning and geometric content

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According to the modern view of geometry, geometrical content is determined by a consistent set of axioms (which define structures implicitly, but do not express propositions) and further articulated through logical deductions. However, neither Bolyai nor Lobachevski, whose investigations of hyperbolic geometry initiated the move towards the modern view, regarded geometry in this fashion. Nor were considerations of consistency the driving force of Beltrami and Klein’s famous ‘models’ of non-Euclidean geometry. Thus, it seems that the modern view of geometry cannot account for the developments in 19th century geometry that led to it. In this paper, some of the background of the emergence of the modern view of geometry is presented. In particular, I will focus on three distinct developments: the work on ‘abstract’ geometry and its relations to Euclidean geometry (Bolyai, Lobachevski, Beltrami, Klein), the realization of duality in projective geometry (Poncelet, Gergonne), and the work on axiomatizations of geometry (Pasch, Hilbert). These developments reveal that two different modes of reasoning were at play in the geometric investigations in the 19th century: An objectival mode, where language

is merely a means to reason about geometric objects, and an linguistic mode, which focuses on the language itself. I will argue that these two modes of reasoning gradually led to a separation of the syntactic (linguistic) and semantic aspects of geometrical reasoning, which can be found in the views expressed by Poincaré and Hilbert at the turn of the 20th century. Finally, I propose two different notions of geometric content that are tied to the two modes of reasoning just mentioned, in order to interpret the various practices of geometry in the 19th century.

## Gödel’s Second Incompleteness Theorem Is Predicate Dependent

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**Abstract.** It is shown that Gödel’s proof of the second incompleteness theorem for formal arithmetic depends on the chosen provability predicate. We come up with an unprovability predicate which is used to construct counterexamples to the second theorem and prove that in the general case the conclusion of the second theorem is not true.

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According to Gödel’s second incompleteness theorem, if the formal Peano arithmetic (PA) is consistent then *Consis*, i.e., the formula

$$\exists x \forall y \neg \text{Prov}(x, y), \quad (*)$$

is unprovable in PA. But the proof of the second theorem is clearly not sufficient to conclude that no other (nonequivalent to Gödel’s *Consis*) formula ‘expressing’ the consistency of PA is provable in PA. The question whether it is possible to construct a decidable formula ‘expressing’ the consistency of formal systems containing PA was taken up by S. Feferman, S. Kleene et al. However, the authors construct their formulas ‘expressing’ the consistency of PA as derivatives of the Gödel formula *Consis* and the provability predicate. But the provability predicate is not the best candidate to ‘express’ the consistency of PA.

The following is a simple consequence of the second theorem showing inadequacy of Gödel’s representation of unprovability:

**Corollary** (we call it Theorem 2+). (1) *If PA is consistent, then, for any formula A, a formula that ‘expresses’ the unprovability of A is unprovable in PA.*

(2) *If PA is  $\omega$ -consistent, then, for any formula A unprovable in PA, a formula that ‘expresses’ the unprovability of A is undecidable in PA.*