Constructive Axiomatic Architecture

for scientific theories

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The Received Conception

A theory is a set of symbolic formulas representing sentences (propositions) related by the relation of logical consequence.

An axiomatic theory is a theory that is generated by a finite set of sentences called axioms (and possibly also a set of axiom schemes) via truth-preserving logical inferences (syntactically represented as syntactic operations on formulas that allow one to built new formulas from some given formulas).

Motivation : The "core" of a theory is concentrated in a short list of "fundamental truths" ; all other theoretical truths are logical consequences of these basic truths.

Logical Consequence (after Tarski 1936)

A is a logical consequence of $B_1, \ldots, B_1$, in symbols

$$B_1, \ldots, B_1 \models A$$

just in case in all interpretations (models) where $B_1, \ldots, B_1$ is true A is true.

Motivation : The relation of logical consequence is objective in the sense that it holds or doesn't hold whether or not one infers A from $B_1, \ldots, B_1$.

Soundness and (semantic) Completeness

If A is syntactically derivable from $B_1, \dots, B_1$, in symbols,

$$B_1, \dots, B_1 \vdash A$$

then

$$B_1, \dots, B_1 \models A$$

.

Soundness is a sine quae non property of formal logical calculus : if a calculus is not sound then its derivations fail to represent logical inferences.

Semantic Completeness : the converse (every logical consequence is syntacticly represented by an inference)

Proofs

Proofs of theorems are logical inferences from the axioms.
Studying syntactic derivations in an axiomatic theory (by external mathematical means) one is doing Proof-theory.

Historical Origins of the Received Conception

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- A. Tarski : Intro to the Logic and the Methodology of Deductive Sciences (1933 Polish, 1937 German, 1941-English)

Meta-mathematical focus

Epistemic completeness : a theory is called epistemically complete when all its theorems are known or knowable.

Background : *Ignoramibusstreit* (the debate about unknowable) started back in 1872 by Emil du Bois-Reymond ;

- Hilbert 1900, Paris : In der Mathematik gibt es kein Ignoramibus
- Hilbert 1930, Königsberg : Wir müssen wissen, wir werden wissen

The original “syntactic” version : Mathematics

Axiomatic Set theory and Proof Theory (FOM) : a meta-mathematical study ;

No role in the representation and transmission of mathematical results. An impact of mathematical education (the informal version of Hilbert-style Axiomatic Method in the spirit of Grundlangen 1899). Recall of the meta-mathematical focus of Hilbert Program.

The original "syntactic" version : Science

Attempts in 1930-ies to apply AM in physical theories (Mathematische Grundlagen der Quantenmechanik 1932 by von Neumann)

and theoretical biology (Axiomatic Method in Biology 1937 by Woodger) ;

later attempts have been marginal. A sever critic from the logical camp for a poor logical standard (Bunge 1967, 1972).

The Semantic Upgrade : Mathematics

Bourbaki (1938 -) :

Lagrange theorem is not a logical consequence of axioms of Set theory !

A syntactic upgrade : Semantic signatures.

The Semantic Upgrade : Science

The "semantic view" aka "non-statement view" of scientific theories : Patrick Suppes and et. al. (including German Structuralism of Stegmüller and Balzer) since 1953.

Practical Impact of the Semantic Upgrade

Mathematics : a better but still very controversial performance of the Bourbaki-style approach in mathematics. The "New Math" in the whole saga of Bourbaki-inspired reforms of the elementary mathematical education across the globe.

Natural Sciences : Only in the Philosophy of Science (where Suppes-style reconstructions of scientific theories are used in a logical analysis of science made for philosophical (rather than scientific) purposes.

Computer Science : Formal Ontology and Knowledge Representation (called by this name by a conceptual mistake)

The Upshot on the Performance of the Received AM

Meta-theoretical tasks are performed but representational tasks are not.

Representational tasks are more basic : unless a formal system represents a mathematical or scientific theory adequately its meta-theoretical properties are irrelevant or very weakly relevant.

The semantic upgrade does not essentially improve the situation.

What is wrong with the Received Conception ?

The very notion of theory as a set of statements related by the logical consequence relation. The "non-statement" nature of scientific theories needs to be made more precise and explicit also in a positive way. What is a theory if not a set of statement ?

Motivations

- Euclid 'Elements'. The role of geometrical constructions in proofs. Euclid's 'Postulates' which are rules with a non-logical semantics, which regulate geometrical constructions "by the ruler and compass" ;
- Hilbert and Bernays's 1934 remarks, in which the authors point to limits of their standard "existential" axiomatic method and discuss the possibility of a more general axiomatic approach ;
- Cassirer's emphasis of the epistemic role of objecthood in the foundations of mathematics (teaser)

Motivations (contd)

- The formal *Calculus of Problems* proposed in 1932 by Andrey Nikolayevitch Kolmogorov ;
- Ideas of Vladimir Alexandrovitch Smirnov concerning the possibility of using the "genetic method" for building scientific theories, which this author developed beginning in the early 1960s
- The Univalent Foundations of mathematics by Vladimir Voevodsky, which involve a non-standard Gentzen-style formal architecture, that effectively combines logical inferences with geometrical (to wit homotopical) constructions (the tope one).

More Theoretical Background : Two Axiomatic styles

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- Hilbert-style : many axioms and few rules (usually just one or two : modus ponens and substitution)
- Gentzen-style : many rules and no axioms : ex. Natural Deduction, Euclid's geometry (perhaps not pure), HoTT (plus Univalence Axiom), Cubical Type Theory (pure).

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- (terminological) Following Aristotle I call rules axioms realising risks of confusion.
- Gentzen-style formal systems are more apt to computational implementations than the Hilbert-style system.
- A general theory of relationships between Hilbert- and Gentzen-style systems waits to be developed. The Deduction Theorem (proved for a number of logical calculi) is a relevant piece of knowledge but it doesn't give answers to all relevant questions.

What is aTheory ? (a new answer)

By a *theory* I understand a fragment of theoretical *knowledge* construed and represented as a system of theoretical *objects* supplied with *rules* for (human) manipulations with these objects.

What is aTheory ? (a new answer)

By an *axiomatic theory* I understand a theory represented with a small number of distinguished *elementary objects* and *elementary rules*, which allow one to generate new objects from given objects and formulate new rules on the basis of given rules. On the syntactic level this procedure is referred to as *derivation* ; I don't assume that its intended semantics is logical.

In special semantic contexts, such derivations can be called *deductions*, *productions*, *constructions*, and by other names. We use the term "object" here in the most neutral and general sense, which covers not only things like points, spaces, physical particles and living organisms, but also things like symbols, formulas, propositions and judgements

Objects and Rules

theory	elementary objects	elementary rules
Euclid	points and equalities	Postulates and Axioms
Hilbert 1899	geometrical axioms	unspecified logical rules
Hilbert 1934	geometrical and logical axioms	modus ponens and subs
MLTT	atomic and base types	MLTT rules
HoTT/UF	point, nat. numbers, UA	MLTT rules
CTT/UF	point, nat. numbers	CTT rules

How propositional and non-propositional theoretic elements relate to each other ?

General answer : The latter serve as evidences (aka witnesses, proofs) for the former.

Commentary : It is not sufficient to rely on intuition and a good luck for finding an appropriate model for a given theory. It is not sufficient to rely on known models and build theories that describe (satisfy) these models. Models need to be built along with theories according to certain formal rules. The choice of such rules is a new and so far very little explored dimension of the "axiomatic freedom" (Hilbert-Detlefsen).

Constructive view of theories : experiments

$$T \xrightarrow{m} M_T \xrightarrow{t} M_{TE} \xrightarrow{e} M_E$$

where the first arrow represents modelling, the second represents the design of a thought-experiment, and the third represents the experimental design. As the above diagram suggests, the language of T remains in this case interpretable in experiments designed for testing this theory.

Scientific Methods

Methods are central in mathematics and science (recall of Descartes). Scientific methods do not reduce to heuristics. Scientific methods play an essential role in justification of scientific theories. This does not limit to logical methods : think of geometrical constructions in Euclid and about physical experiments that justify theories in this discipline.

Logical methods apply in science only along with extra-logical methods such as methods of performing physical experiments. Extra-logical methods are proper elements of their corresponding theories.

Scientific Methods (contd)

The recieved Hilbert-style axiomatic approach in the representation of scientific theories leaves extra-logical methods aside the theory proper to the “context of discovery” of this theory. This is a serious epistemological mistake. This mistake is not fixed with the semantic upgrade of Hilbert’s method (see above).

We fix this mistake with new formal means and give extra-logical scientific methods a place in the “context of justification” of corresponding theories.

Formal Representation of Scientific Methods

Syntactically scientific methods are representable with Gentzen-style (rule-based) formal calculi. HoTT/UF provides a convenient semantical framework where logical methods are integrated with the non-logical ones (i.e. methods of performing extra-logical mathematical constructions similar to Euclid's geometrical constructions.)

Further research

Provide examples from Science where this approach works. Bridge Science and Logic rather than reserve for Logic a pure philosophical use (in Metaphysics, Analytic Philosophy, Neo-Scholastics, etc)

Thank You !