Voevodsky's Unachieved Project

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NYC Category Theory Seminar

21 October 2020

Vladimir Voevodsky (1966-2017) made important contributions into two different branches of mathematics: the Algebraic Geometry and the Foundations of Mathematics.

He also spent a significant part of his time and effort working in Applied Mathematics and, more specifically, in the Mathematical Biology. This project remained unachieved.

Interview with Roman Mikhailov, 2012

Since Fall 1997 I realised that my main contribution into Motive theory and Motive Cohomology was already accomplished. Since then I was very consciously and actively looking for ... a theme to work on [...]. [C]onsidering tendencies of development of mathematics as a science I realised that [...] mathematics is at the edge of crisis, more precisely, two crises.

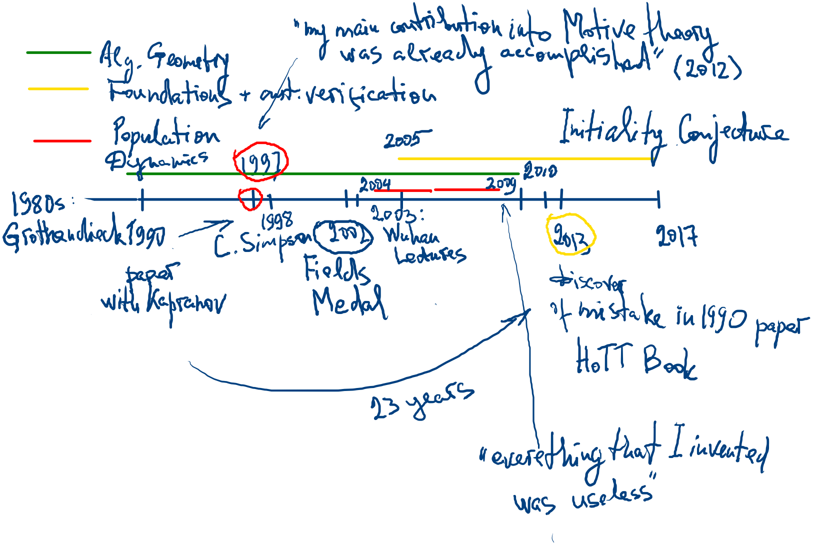
The first crisis concerns the gap between "pure" and applied mathematics. It is clear that sooner or later there will arise the question of why the society should pay money to people, who occupy themselves with things having no practical application.

The second crisis, which is less evident, concerns the fact that mathematics becomes very complex. As a consequence, once again, sooner or later mathematical papers will become too difficult for a detailed checking, and there will begin the process of accumulation of errors.

I decided to do something in order to prevent these crises. In the first case that meant to find an applied task, which would require for its solution methods of pure mathematics developed during the last years or at least during the last decades. [...]

[I] was looking for open problems, which would be interesting for me, where I could apply today's mathematics. Finally, I chose — as I now understand wrongly — the problem of reconstruction of history of populations [of living organisms] on the basis of their present genetic constitution. I worked on this problem about two years and finally realised in 2009 that everything that I invented was useless. [...]

Timeline


"my main contribution into Motive theory was already accomplished" (2012)
Alg. Geometry
Foundations + aut. verification
Population Dynamics
1997
2005
Initiality Conjecture
2010
2004
2009
1980s:
Grothendieck 1990
paper with Kapranov
C. Simpson
1998
2002
Fields Medal
2003:
Wuhan Lectures
2013
2017
discover of mistake in 1990 paper
HoTT Book
23 years
"everething that I invented was useless"

What is most important for mathematics in the near future? Wuhan U, Dec. 2003

There are two most urgent needs in today's mathematics:

1. (A) To build a computerised library of mathematical knowledge, i.e., a computerised version of Bourbaki's *Elements*;
2. (B) To bridge Pure and Applied Mathematics.

How to do that?

(A) We should gradually move from a hyperlinked mathematical text to a mathematical text verifiable with computer.

(B) We discovered very fundamental classes of new objects including categories, sheaves, cohomology, simplicial sets. They may turn out to be as important in science as algebraic groups. But presently we don't use them for solving problems outside the Pure Mathematics.

Why todays maths is not applied?

One reason can be sociological. Only few people have a profound knowledge both of modern mathematics and of some other research field where an application of modern mathematics can be possible.

Another reason concerns the current scientific policies. In order to apply an abstract mathematical theory to a concrete practical problem one needs, first of all, to generalise this problem and abstract away the intuition associated with this problem. But the current funding policies favour rather fast solutions of concrete practical problems such as, for example, designing "the billion dollar drug".

How to proceed?

In order to apply mathematics to a given problem outside mathematics one should begin with the opposite move. Instead of trying to concentrate on future applications of a mathematical theory to the real life, one should abstract yourself from the real life and look at the given problem as a formal game or puzzle. This is a reason why new mathematics too often strikes one, wrongly, as what moves away from real-world problems.

So the only reasonable policy in mathematical research and in science in general is to support one's curiosity and one's sense of beauty in science.

Brief Comments

Compare:

- Vladimir's family and school background in Physics and Chemistry;
- Philosophy of Mathematics in the Soviet Dialectical Materialism (Ruzavin);
- I.M. Gelfand work in the Mathematical Biology: "There is only one thing which is more unreasonable than the unreasonable effectiveness of mathematics in physics, and this is the unreasonable ineffectiveness of mathematics in biology "

Four Layers of Today's Mathematics

1. Elementary Mathematics: Pythagoras theorem, Quadratic Equations, etc.; it emerged more than 1000 years ago;
2. "Higher" Mathematics : Integral and Differential Calculus, Differential Equations, Probability theory [17-18 c.];
3. Modern Mathematics: Modern Algebra (Galois theory, Group theory), Basic Topology, Logic (including Gödel Incompleteness theorems) and Set theory [c. 1850-1950];
4. Synthetic Mathematics: Representation theory, Algebraic Geometry, Homotopy theory (in particular the Motivic Homotopy theory), Differential Topology [since 1950].

Four Layers of Today's Mathematics (cntd)

- Elementary Mathematics is integrated into the everyday life;
- "Higher" Mathematics is integrated into most sciences;
- Modern Mathematics is integrated into *some* sciences;
- Synthetic Mathematics is very poorly integrated (if at all).

Mathematics and the Outside World (AMS-India meeting Bangalore, Dec. 2003

Mathematics is an integral — albeit very special — part of general problem-solving activity, which by itself is a pre-scientific human condition.

Various practical problems, which are conceptualised and approached with the common aka *conventional* thinking are more effectively solved via the *mathematical modelling* aka applied mathematics, which in its turn gives rise to the pure mathematics.

The pure mathematics grows with solving such *external* problems, which come via the mathematical modelling, and also with formulating and solving its own *internal* problems (in the form of proving mathematical *conjectures*).

Flow of Problems and Solutions

$$\begin{tikzcd}
\text{Conventional Thinking} \arrow[d, shift right] \\
\text{Math Modelling} \arrow[u, shift right] \arrow[d, shift right, "1"'] \\
\text{Pure Math} \arrow[u, shift right, "2"'] \arrow[loop below, "\boxed{\textit{conjectures}}"]
\end{tikzcd}$$

Flow of Problems and Solutions (cntd)

Over the last few decades the situation [as described above] was getting more and more out of balance. Arrows 1 and 2 shown at the diagram, which connect pure and applied mathematics, were weakening.

A weak incoming flow of external problems restrains the internal development of pure mathematics. A weak outgoing flow of useful solution restrains the support of mathematics provided by the Society.

Breakdown of arrow 2 means eventually no salary for mathematicians.

Breakdown of arrow 1 means eventually no new ideas in mathematics.

What we should do to improve the situation

Change the existing pattern of using computer technologies in the general problem-solving

The existing pattern

$$
\begin{tikzcd}
\text{Conventional Thinking} \arrow[d, shift right] \\
\text{Computer Modelling} \arrow[u, shift right] \arrow[d, shift right] \\
\text{Math Modelling (MM)} \arrow[u, shift right] \arrow[d, shift right] \\
\text{Pure Math (PM)} \arrow[u, shift right] \arrow[loop below, "\boxed{\textit{conjectures}}"]
\end{tikzcd}
$$

The existing pattern (cntd)

The flow of problems down to the "mathematical modelling" layer is filtered through the "computer modelling layer".

As a result the "mathematical modelling" layer, and as a consequence also the "pure mathematics" layer, receive less problems than they used to receive before the rise of computer technologies.

This particularly affects today's abstract mathematics. Problems, which pass through the filter, are formulated in the old-style language of variables and analytic functions, while the language of today's abstract mathematics is the Set theory.

Double Translation of Problems

$$
\begin{tikzcd}
\text{Computer Model} \arrow[d] \arrow[dr, dashed] & \\
\text{Analytic language of MM} \arrow[r] & \text{Set-theoretic language of PM}
\end{tikzcd}
$$

The new pattern

$$\begin{array}{c} \text{Computer Model} \\ \downarrow \\ \text{Set-theoretic Model} \\ \textit{sometimes} \downarrow \\ \text{Analytic Model} \end{array}$$

Implementation

In order to implement this new scheme we need, in particular, to reformulate fundamental and applied scientific theories in the language of today's abstract mathematics, viz., the set-theoretic language.

For this end we need to specify for each theory a notion of basic *unit* and then consider sets of such units.

Sciences and Their [Ontological] Units

Science	**Unit**
Population Biology and Demography	Individuals (individual
Financial Mathematics	Companies
Political Science	Voters
Particles Physics	Particles
Population Genetics	Genes
Future Theoretical Chemistry, which will be able to account for individual molecules	Molecules

The most important task for mathematicians is to produce examples that demonstrate the effectiveness of this approach. ■

Set theory as a mathematical foundations for Science

- Georg Cantor considered applications of his Set theory in Biology (following Riemann; letter to Mittag-Lefler Sept. 1884);
- "semantic view of theories": Bourbaki-style approach in science: Patrick Suppes et al. since 1953
- formal ontologies in Analytic Metaphysics and CS.

Set theory as a mathematical foundations for Science (cntd)

Voevodsky's *Singletons* (2008-2009, 94 pages) are set-based. "Singletons" are "identical age-less individuals", i.e., idealised living organisms. The failure of this project is one more evidence that the set-based ontological approach in science does not work.

Cf. Lawvere 1970: "a 'set theory' ... should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets, which do in fact develop along such parameters." (He means here Topos theory.)

There is a large body of works aiming at applications of Category theory (flat and higher categories) in theoretical Physics: Lawvere, Baez, Schreiber (QFT) et al. Interestingly, Vladimir did not follow this path but looked for applications of recent "synthetic" mathematics in empirically-oriented (data-driven?) rather than theoretical scientific studies.

New mathematical foundations for Science?

It appears to me very plausible that Voevodsky's dismissal of his Population Dynamics research in 2099 is related to his new ideas about Foundations of Mathematics, viz., the Univalent Foundations (even if these new foundations were motivated differently). See the timeline.

Is UF appropriate as a mathematical foundation for science?

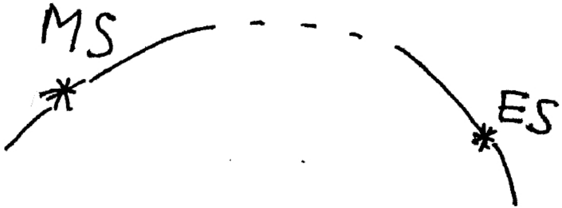
UF provides for the wanted shortcut from Computer Models to Mathematical Models (double translation problem):

$$\begin{tikzcd}
\text{Computer Model} \arrow[d] \arrow[dr, dashed] & \\
\text{Analytic language of MM} \arrow[r] & \text{Set-theoretic language of PM}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Computer Model} \arrow[d] \\
\text{UF Model} \arrow[d, "\textit{sometimes}"'] \\
\text{Analytic Model}
\end{tikzcd}$$

Is UF appropriate as a mathematical foundation for science? (cntd)

UF is intuitive (in low dimensions). Cassirer 1907: "The [Kantian] principle according to which our [mathematical] concepts should be sourced in intuition means that they should be sourced in the mathematical physics and should prove effective in this field."



Is UF appropriate as a mathematical foundation for science? (cntd)

UF supports extra-propositional constructions (of *h*-level $>$ (-1)). A scientific theory does not reduce to a set of true sentences. It also includes experimental and observational methods that allow one to verify such sentences.

Think of thought-experiments and their implementations in form of physical experiments and (prepared, theory-laden) observations.

Is UF appropriate as a mathematical foundation for science? (cntd)

A related example: TDA and its tentative applications in the Brain research.

Axiomatic Architecture of Scientific Theories

http://philsci-archive.pitt.edu/17600/

Abstract: The received concepts of axiomatic theory and axiomatic method, which stem from David Hilbert, need a systematic revision in view of more recent mathematical and scientific axiomatic practices, which do not fully follow in Hilbert's steps and re-establish some older historical patterns of axiomatic thinking in unexpected new forms. In this work I motivate, formulate and justify such a revised concept of axiomatic theory, which for a variety of reasons I call constructive, and then argue that it can better serve as a formal representational tool in mathematics and science than the received concept.

THANK YOU!