Euclid's Geometry as a Gentzen-style Theory : Euclid and Today's Mathematical Practice

Section: History and Philosophy of Mathematics

Andrei Rodin

Institute of Philosophy RAS

andrei@philomatica.org

Abstract:

As David Hilbert later acknowledged himself, his novel axiomatic presentation of Euclidean geometry in his 1899 *Grundlangen der Geometrie* not only provided for a higher degree of logical rigor but also left wholly aside an interesting and possibly valuable aspect of Euclid's theory that Hilbert called “genetic” or “constructive”. As it has been earlier remarked by leading Euclid scholars including Ian Mueller the “postulates” and “axioms” of Euclid's 'Elements' are not statements but rules that validate certain geometrical constructions and certain propositional inferences. In modern logical terms Euclid's geometry in its original form qualifies not as a Hilbert-style axiomatic theory but as a Gentzen-style, i.e., a rule-based theory.

In this paper we survey the rule-based structure of Euclid's geometrical reasoning and show its relevance in today's mathematical practice. Finally, we point to Univalent Foundations of mathematics as a recent attempt of building mathematical theories in Gentzen-style and consider some common features of this approach and Euclid's traditional approach.

Keywords: Euclid, Hilbert-style and Gentzen-style theories, axiomatic method

1) Euclid's geometry

By Euclid's geometry hereafter we understand the mathematical theory exposed in first four Books of Euclid's 'Elements' [1], not the Euclidean geometry construed as a modern axiomatic theory. The modern conception of axiomatic theory has been strongly influenced by David Hilbert's 'Foundations of Geometry' first published in 1899 where the author presented the elementary Euclidean geometry in a novel axiomatic form [2]. We fully endorse the received view according to which Hilbert's contribution to the 20th century axiomatic mathematics is of outmost importance. Nevertheless we claim that Euclid's original axiomatic approach has some interesting and useful features, which are wholly neglected in Hilbert's approach. As we shall now see the axiomatic architecture of Euclid's geometry drastically differs from the received Hilbert-style axiomatic architecture of modern Euclidean geometry.

Euclid's geometry is based on 5 Common Notions aka Axioms, 5 Postulates and a number of Definitions. Postulates 1-3 are as follows (verbatim after [1]):

[P1:] to draw a straight-line from any point to any point.

[P2:] to produce a finite straight-line continuously in a straight-line.

[P3:] to draw a circle with any center and radius.

As they stand the three Postulates are not propositions and admit no truth-values.
The non-propositional character of Euclid's Postulates has been earlier noticed by I. Mueller [3], A. Szabo [4], D. Macbeth [5] and some other scholars.
Postulates 1-3 can be best described as schematic rules that specify basic operations, which take some geometrical objects as input and produce some other geometrical objects as output.
These operations are partly composable in the obvious way: the output of P1-operation is used as input for P2- and P3-operations.

The first three Common Notions have the same schematic character but unlike the Postulates they apply to propositions of particular form, not to geometrical construction. Consider Common Notion (Axiom) 1 that reads:

[A1] : Things equal to the same thing are also equal to each other.

Even if [A1] admits a formalization in a propositional form

[(A=C) & (B=C)] --> A=C

an analysis of the 'Elements' suggests that Euclid uses [A1] as a schematic inference rule but not as an assumption [3]:

$$\frac{A=C,\ \ B=C}{A=B}$$

This rule validates inferences of new propositions from given propositions but it doesn't qualify as logical in the usual sense because it applies only to propositions of form X=Y (where “things” X, Y are either geometrical magnitudes or natural numbers) but not to all propositions indiscriminately

There are direct textual evidences that Aristotle used the mathematical Common Notions (that were known before Euclid composed his 'Elements') as prototypes of his logical principles that he called “axioms”. The modern use of term “axiom” as a name of Euclid's Common Notions stems from the Aristotelian tradition [6].

Thus the foundation of Euclid's geometry comprises two sets of rules: one for geometrical constructions (Postulates) and the other for propositions (Axioms). Correspondingly, the main content of Euclid's geometry comprises two sorts of units: Problems and Theorems. Problems can be described as derived rules for constructing complex (as opposed to basic) geometrical constructions, while Euclid's Theorems have the familiar propositional form. Crucially, the two sorts of contents are not independent but form a joint deductive order: Problems involve proofs that the performed constructions achieve their announced goals while Theorems apply complex geometrical constructions in their proofs. This explains why Problems and Theorems in Euclid share the same formal structure.

2) Euclidean approaches in today's mathematics

It may appear that the specific features of Euclid's geometry highlighted above are archaic and can be only of a purely historical interest. This is not the case. Formal logical calculi, which are rule-based rather than axiom-based, have been first proposed by Gerhard Gentzen in 1930-ies and since then thoroughly studied and further developed. Until recently the Gentzen-style rule-based formal approach, unlike the standard Hilbert-style axiomatic approach, had little or no use and impact outside of the pure logical studies. This situation is now progressively changing; an important pragmatic motivation behind this change is the fact that Gentzen-style rule-based formal calculi, generally, can be implemented on computer easier and in a more straightforward manner than Hilbert-style systems. The notion of Curry-Howard correspondence, which plays an important role both in logic and in the theoretical Computer Science, points to a common formal structure of constructive logical reasoning and computing; the structure of Euclid's geometrical reasoning that combines rule-based geometrical constructions with rule-based propositional inferences is a form of such correspondence.

Particularly interesting in the given context is the Homotopy Type theory (HoTT) emerged in mid-2000-ies as the basis of new project in foundations of mathematics named by Vladimir Voevodsky Univalent Foundations [7]. The syntactic core of HoTT is a Gentzen-style type calculus with dependent types known after the name of its inventor as Martin-Löf Type theory (MLTT). As it has been observed by Awodey and Voevodsky (independently) MLTT admits an unintended interpretation in terms of Homotopy theory. The homotopical interpretation makes evident a distinction between different kind of types that earlier has been left unnoticed: not all but only some types (namely, those that have at most a single term) are naturally identified with propositions, some other types are identified with sets while the remaining "higher" types are identified with more complex spatial-like (geometrical) entities. In this way HoTT justifies the idea of rule-based non-propositional construction, which in a different form is found in Euclid, and also explains why and how such non-propositional constructions have a logical impact at the propositional level.

3) Conclusion

It is sometimes said that Euclid's 'Elements' have been widely used as a standard geometry textbook until the invention of non-Euclidean geometries in the 19^{th} century or even until 1899 when Hilbert published his 'Foundations of Geometry' and showed how the Euclidean geometry can be construed in a modern way. This claim is not historically accurate because what mathematicians of older generations called by the name of Euclid was typically a modernized contemporary version of the 'Elements', not the 'Elements' in its original form, which is better known today thanks to significant efforts made by J.L. Heiberg and other historians. Hilbert's 'Foundations' is but one important work in the long series of works aiming at rewriting Euclid in a new way. There is no reason to assume that with Hilbert's 'Foundations' the tradition of rewriting Euclid has its final and definite achievement. Recent developments in logic and foundations of mathematics provide a novel perspective on Euclid highlighting some features of his mathematics that earlier seemed to be archaic and theoretically insignificant. However developing a novel axiomatic approach in the elementary geometry that could qualify as the 'Elements' of the 21^{st} century still remains an open problem.

Bibliography:

[1] Euclid, Elements. English translation by Richard Fitzpatrick, lulu.com 2011

[2] D. Hilbert, Grundlagen der Geometrie, Leipzig 1899

[3] Mueller, I., 'Greek Mathematics and Greek Logic', Ancient Logic and Its Modern Interpretations, Springer 1974, pp. 35-70

[4] A. Szabo, Beginnings of Greek Mathematics, Reidel Publishing Company, 1978

[5], D. Macbeth, Realizing Reason, Oxford University Press, 2014

[6] A. Rodin, Axiomatic Method and Category Theory, Springer 2014

[7], Univalent Foundations Group, Homotopy Type Theory: Univalent Foundations of Mathematics, Institute for Advanced Study (Princeton), 2013