Mic Detlefsen on Frege-Hilbert Controversy

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Sources :

- M. Detlefsen, Hilbert Program, Springer, Synthese Library, **1986**
- M. Detlefsen, On the Motives for Proof Theory, in : H. Wansing (ed.), Dag Prawitz on Proof and Meaning, Springer **2015**, pp. 121-146

Hilbert 1899

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,.. ; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . [..] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between" ; "parallel", "congruent", "continuous", etc.

Frege 1903 adversus Hilbert 1899

From the fact that the pseudo-axioms do not express thoughts it [...] follows that they cannot be premises of an inference-chain. Of course, one really cannot call propositions — groups of audible or visible signs — premises anyway, but only the thoughts expressed by them. Now in the case of the pseudo-axioms, there are no thoughts at all, and consequently no premises. Therefore when it appears that Mr. Hilbert nevertheless does use his axioms as premises of inferences and apparently bases proofs on them, these can be inferences and proofs in appearance only. (quoted in Detlefsen (1986))

Hilbert to Frege c. 1900

You say that my concepts, e.g. "point", "between", are not unequivocally fixed [...]. But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases.

Detlefsen 2015

Frege and Hilbert did not resolve, and scarcely even addressed, the differences between their respective conceptions of axiomatization. Partly too, though, it was because, at the time or their correspondence, Hilberts comprehension of his proof-theoretic alternative was rudimentary and lacking in substantive detail. When Frege requested a more detailed description and an example, Hilbert was not in a position to provide them, and this essentially ended their correspondence. [The emphasis is mine : A.R.]

Detlefsen 1986

Hilbertian instrumentalist does not contend that noesis with respect to a given real proposition is brought about purely as a result of the formal activity involved in constructing an ideal [i.e., formal syntactic] proof for it. Rather, he holds that such formal activity becomes noetic and significant by being contentually assessed at the level of metamathematics.

Detlefsen 1986 (contd)

This contentual metamathematical assessment, constituting as it does a genuine proof of a real mathematical proposition, can then be used as an epistemic replacement for real mathematical proofs of that proposition without violating Frege's strictures.

For this reason, we will often refer to Hilbert's instrumentalism as the "Metamathematical Replacement Strategy". [MMRS]

Detlefsen 1986

We must now say how it is that Hilbert's Metamathematical Replacement Strategy provides for [...] explaining how noesis formed via the Hilbertian mechanism might aspire to the higher forms of noesis. On the Fregean model this is accomplished by dint of the fact that the constituent premises and inferences of the genuine proof that results from the semantical interpretation of a computation are supposed to provide an evidensory basis for the proposition expressed by its conclusion.

Detlefsen 1986 (contd)

On the Hilbertian model, the constituent premises and inferences of the genuine metamathematical proof that arises from the metamathematical evaluation of the computation can, by reasoning parallel to that of the Fregean case, be seen as providing an evidensory basis for its (i.e., the metamathematical proofs) conclusion. And since the conclusion of that proof is just the proposition expressed by the terminal formula of the computation, it follows that the metamathematical proof provided by the Hilbertian model is capable of founding the higher forms of noesis regarding the conclusion of a computation.

Dilution Problem (1986)

[T]he Dilution Problem, as we shall present it, shall be primarily concerned with the comparative strength of contentual mathematical proof and its proposed metamathematical replacement.

It is the Dilution Problem that moves the Hilbertian instrumentalist to embrace finitism.

Equivalency Thesis (1986)

The really crucial element of [Hilberts]the argument is not the attribution of this or that particular character to number-theoretic reasoning, but rather the essential identification of its character with that of proof-theoretic or metamathematical reasoning. This Equivalency Thesis is asserted by Hilbert repeatedly .

Rigour and Truth (30 years later : 2015)

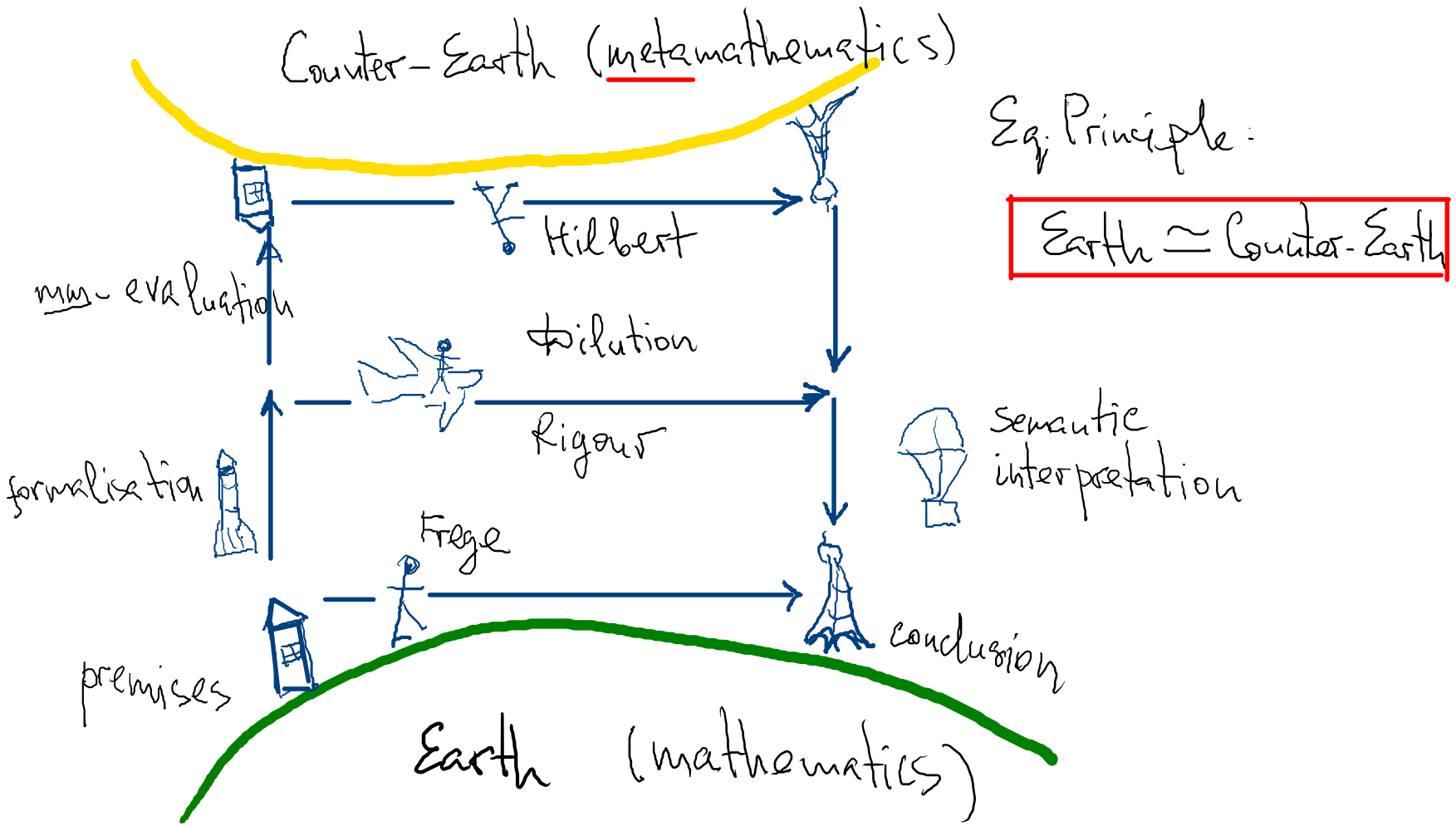
An axiom cannot be exactly specified by giving a sentence that expresses it, that is, by giving a sentence which, interpreted in a certain way, semantically signifies it. Rather, it can be exactly and completely specified only by giving a sentence which is it, or more accurately, a sentence which exhibits or externally exemplifies it.

Rigour and Truth (30 years later : 2015)

If, therefore, the axioms of an axiomatic theory are to be exactly and completely specifiable, they must be sentences or formulae, not propositions or other semantical contents.

If this is essentially right, then, abstract axiomatization is to be seen as aiming at the separation of contents from proof not only for purposes of deduction, but also for purposes of specification.

Reasoning as a Journey



Is MMRS exotic ? Pro :

A special emphasis on metamathemtical questions in foundations of mathematics : consistency, epistemic completeness, ...

Hilbert's public involvement into the *Ignoramibusstreit* (Emil du Bois-Reymond since 1872, see McCarty :2005)

- 1900 : [I]n der Mathematik gibt es kein Ignorabimus
- 1930 : Wir müssen wissen, wir werden wissen !

Is MMRS exotic ? Contra : Geometry

$0<k<1$
$0<l<1$

$$\begin{cases} a+b+c=0 \\ a+b/2=d \\ c+a/2=e \\ lk-dl=c \end{cases}$$

$\overrightarrow{AB}=a$
$\overrightarrow{BC}=b$
$\overrightarrow{CA}=c$
$\overrightarrow{AD}=d$
$\overrightarrow{CE}=e$

formalisation

$$a\left(l+\frac{k}{2}-1\right)+b\left(k+l/2-1\right)=0$$

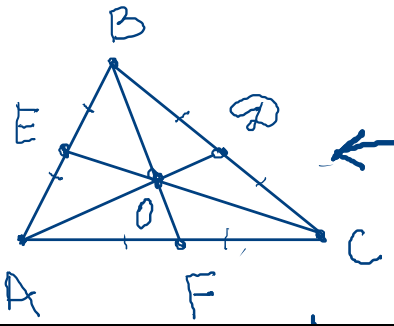
$$\begin{cases} l+\frac{k}{2}=1 \\ k+l/2=1 \end{cases} \qquad l=2/3 \qquad k=2/3$$

interpretation

$$\frac{DO}{AO}=\frac{1}{2}$$

$$\frac{EO}{OC}=\frac{1}{2}$$

$$\frac{FO}{OB}=\frac{1}{2}$$



geometrical proof

Remarks

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- Common spatial intuitions support a plenty of different geometrical theories which, in their turn, admit (interpret) a plenty of different symbolic calculi. The relation between geometrical theories and associated symbolic calculi is not one-to-one but many-to-many. Descartes uses two different geometrical interpretations of formal algebraic product. Think of Grassmann Algebra, Projective Geometric Algebra, Homological Algebra, etc.

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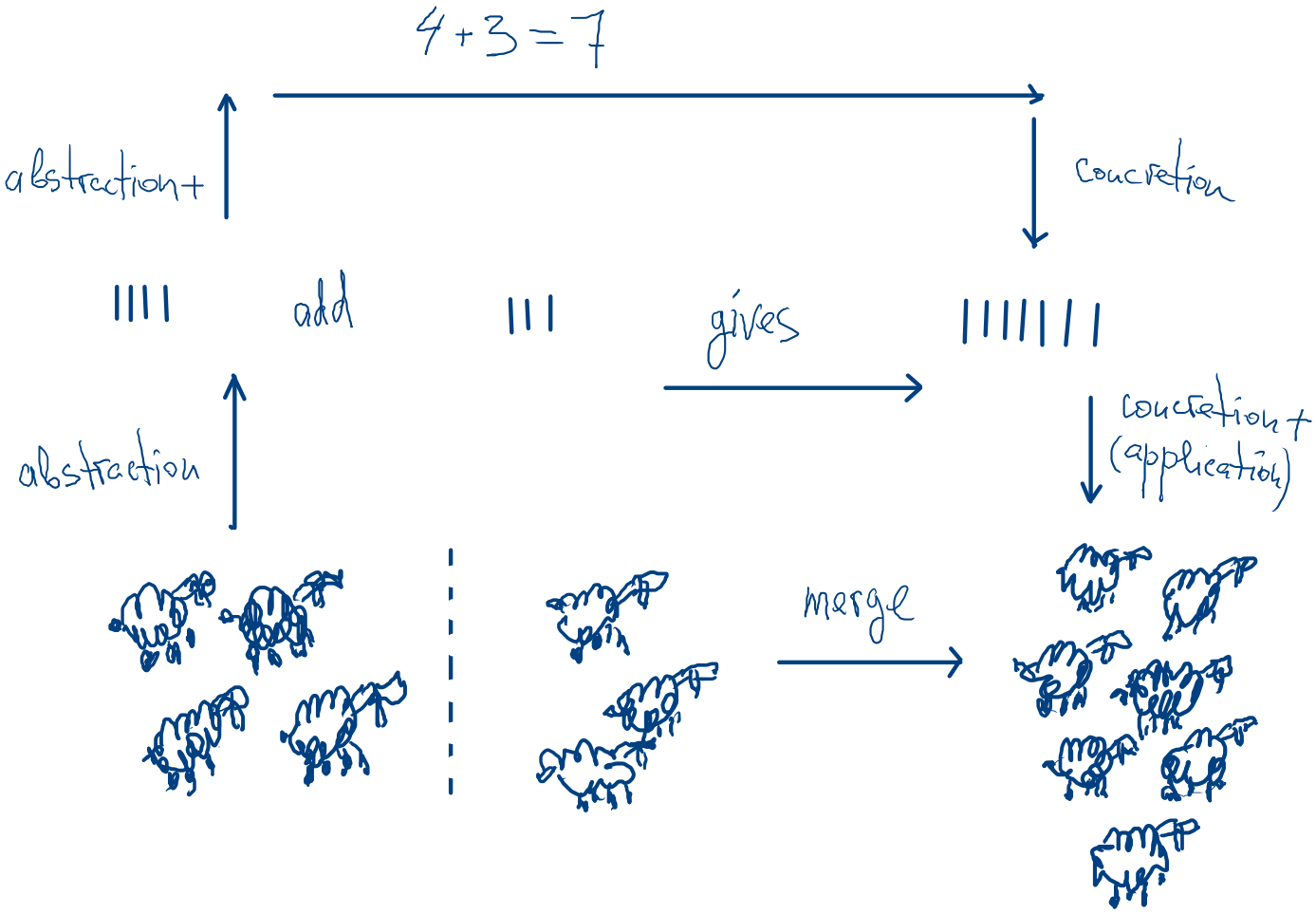
- Geometrically-motivated algebraic calculi are, typically, not separated from their geometrical backgrounds but seen as elements of the corresponding geometric theories. Such theories educate and transform common spatial intuition in various ways. For a recent example consider the case of Topological Data Analysis.

Is MMRS exotic ? Contra : Arithmetic



Remark

In this case it is common to make a sharp epistemic cut between pure arithmetic, on the one hand and its material background (application), on the other hand. It is common to hold that $4+3 = 7$ is true even if some particular material interpretations of this arithmetical sentence prove false.

Continuum Hypothesis (CH)

Set theory is unique among mathematical theories in its reliance on the formal axiomatic approach. The outcome of using MMRS in solving the Continuum Problem is impressive but nevertheless controversial :

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- There is no consensus on whether or not this independence result closes the the Continuum Problem. Arguably, the problem can and should be effectively solved in a different formal/mathematical setting (Hugh Woodin).

Continuum Hypothesis (CH)

Making an epistemic cut between Cantor's "naive" Set-theory and ZF is not quite appropriate. According to the standard rhetoric ZF "makes Cantor's assumptions explicit and clear". But alternative ways to formalise Cantor's Set theory, are equally conceivable. Think of ETCS, the concept of set as a discrete homotopy space in HoTT , etc.

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- Con_{PA} is not derivable in PA (Gödel 1931, the Second Incompleteness Theorem, G2). Standard interpretation : the consistency of arithmetic is not provable in arithmetic. Proving the consistency of arithmetic by stronger means involves a vicious epistemic circle in foundations.
- But Gödel's Con_{PA} is not the only one, and arguably, not the best, candidate for serving as a formal expression of informal PA-consistency statement. Recall Dag Prawitz' notion of General Proof theory and the growing body of works in the Proof-Theoretic Semantics.

The Provability of Consistency (Artemov 2020)

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- represents the informal PA-consistency statement with a scheme Σ of PA-formulas
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- provides a formal prove of scheme Σ in PA.

Remark

The idea of canonical (standard) formalisation (Suppes) of mathematical theories is hardly justified. Mathematical theories, generally, admit multiple formal versions, which are not always equivalent (whatever this may mean). Reciprocally, formal theories, generally, admit multiple informal interpretations. The relation between formal and informal theories is many-to-many like in the case of geometrically motivated algebraic calculi.

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It makes sense to think of ZF-like sets as a special object of study, and distinguish such sets from sets of different sorts.

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- KC : Hales&Flyspeck project team : 2017

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- General reasons why the given computation and a particular result of this computation is a conclusive evidence for the theorem's statement, e.g., soundness of logical calculus
- A survey of symbolic computation (which is not fully available for 2CT and KC)

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- Global Surveyablility : Providing reasons why the wanted proof reduces to certain computation (the informal part of proof)
- Local Surveyablility : Each single elementary step of symbolic computation and each piece of corresponding code are perfectly surveyable by human.
- The local and the global surveys are *disconnected* : the assemblage of surveyed computational steps does not form a global picture. This feature makes computer-assisted proofs opaque and arguably unsatisfactory.

HoTT/UF

HoTT/UF solves this problem by associating certain intuitive geometrical (to wit homotopical) constructions to symbolic computations in MLTT, which also admits a proof-theoretic logical semantics.

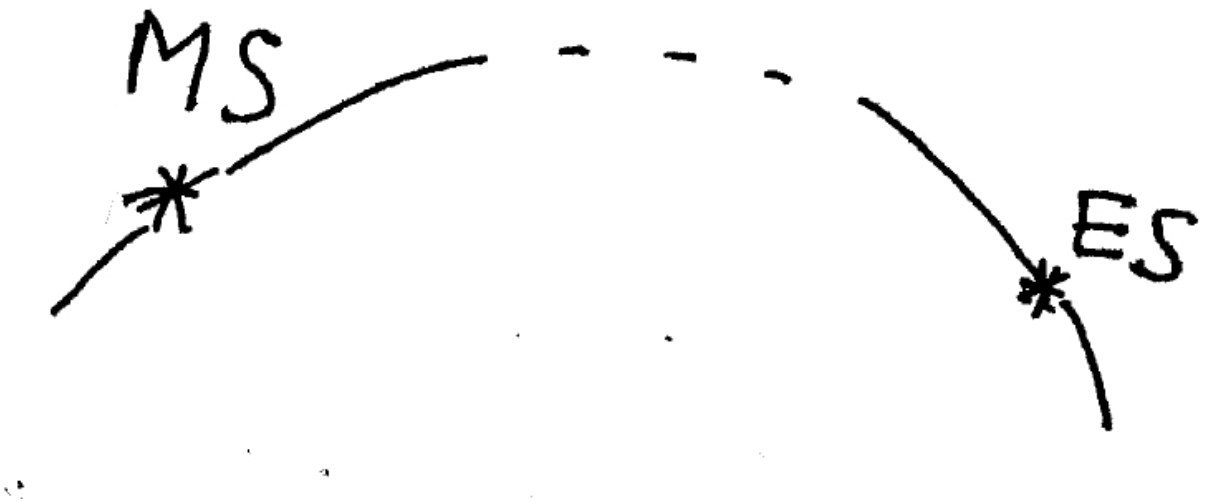
types — — spaces

terms — — points

The logical and the extra-logical geometrical semantics of MLTT combine in a traditional way : geometrical constructions serve as truth-makers of corresponding propositions and thus form judgements.

$a : A$ (a witnesses A) — — point a is in space A

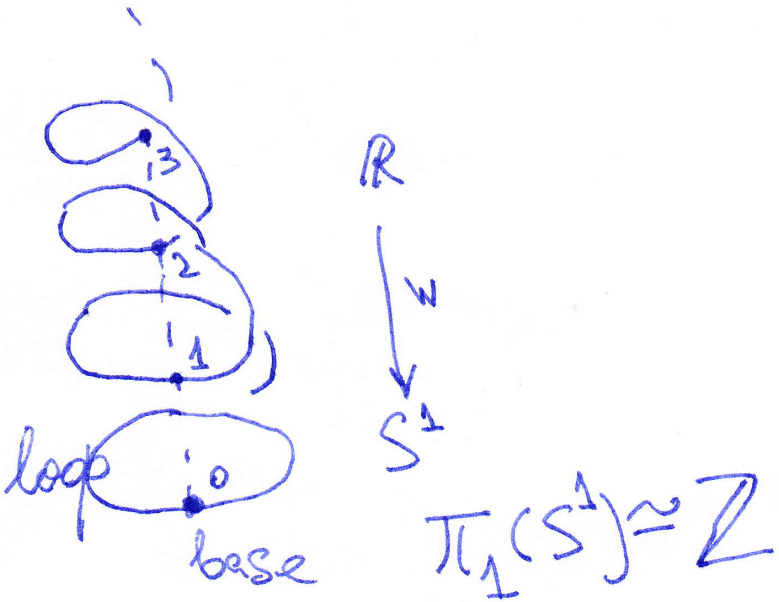
Morning Star is Evening Star



$$path : MS \ =_U \ ES$$

$\pi_1(S^1) \simeq \mathbb{Z}$

Homotopical constructions in HoTT provide for an intuitive grasp of symbolic computation on the intermediate mesoscopic level that supports the desired synthesis of computational steps, which leads to understanding. Ex. : Licata&Shulman :2013



mathematics or metamathematics ?

Where the homotopical interpretation of MLTT belongs ? HoTT is not a model of MLTT in the (set-based) Homotopy theory. It can be thought of as a metamathematical tool for studying the syntax of MLTT. However, it can be equally thought of as a "synthetic" axiomatic Homotopy theory, which can serve as a foundation of other mathematical theories (in UF). This latter optics appears to me more natural.

"[G]ood metamathematics is good mathematics rather than shackles on good mathematics" (Manin 2002)

Conclusion (1) :

MMRS is a special case of a basic mode of mathematical reasoning that can be called the *symbolic computational abstraction*.

Frege's neglect of this basic feature of mathematical reasoning in his logical analysis of mathematics is a mistake.

Hilbert's idea to reduce a mathematical proof to a "finitary" metamathematical syntactic reasoning is not justified either because of the surveyability / understanding problem.

Conclusion (2) :

The symbolic computational abstraction in mathematical reasoning is balanced by the opposite process that can be named the *imaginary concretion*, which is essential, in particular, in applications of mathematics.

Examples : geometrical interpretation of complex numbers, the homotopical interpretation of MLTT in HoTT, TDA.

Conclusion (3) :

The relation between informal mathematical theories / constructions and their formal symbolic computational counterparts is many-to-many : a formal theory admits multiple informal interpretations, and an informal mathematical theory admits multiple formal versions, some of which may be essentially different.

There is no "standard formalisation" of a given (informal) mathematical theory.

Conclusion (4) :

Different formal symbolic logical frameworks implement different normative ideas about mathematics and capture different aspects of mathematical practice.

Metamathematical results concerning such frameworks do not resolve philosophical controversies about logic and mathematics but enrich a theoretical context for developing philosophical ideas and arguments. Taking a metamathematical result (e.g. Gödel's Second Incompleteness theorem) for an epistemological claim (e.g., that a consistent theory which includes arithmetic cannot proof its own consistency) is a mistake.

Thank You !