Axiomatic and Genetic Methods of Concept- and Theory-Building: An Attempt of Synthesis

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Bunge 1972

"In his epoch-making book [von Neumann, *Mathematical Foundations of Quantum Mechanics*, 1932]) [...] von Neumann is wrongly supposed to have laid down the axiomatic foundations of quantum mechanics. As a matter of fact his exposition lacks all the characteristics of modern axiomatics [...]. Yet for some strange reason it passes for a model of physical axiomatics."

Remark:

There is a persisting gap between logicians' and physicists' concepts of a theory.

The idea of logical and metaphysical foundations of natural science, has been banished from Physics in the early 17th century by Galileo and his followers as a part of the traditional Aristotelian background of their contemporary physics. Logic remains under suspicion among physicists ever since.

Bunge 1972 (compare to Tarski's 1941)

"There is a single theory that starts from scratch: mathematical logic [. . .]. More precisely, the least a mathematical or a scientific theory takes for granted is the so-called ordinary (two-valued) predicate calculus [. . .]."

What justifies theories of mathematical logic and, in particular, the predicate calculus? No *physical* justification available. (Should one have one?)

Attempted solution: the semantic (aka non-statement) view of theories

Suppes and others since late 1950ies: A theory is a *class of models* but not an axiom system, nor its deductive closure ("semantic view").

Suppes 2002: Term "model" is used in logic and science similarly in spite of apparent differences. The Tarskian notion of set-theoretic formal semantics provides a universal formal semantics for science.

Bourbaki-style theories

Practical motivation: application of Bourbaki-style formal "semantic" technique beyond the pure mathematics.

Using this formal technique allows one to ignore syntactic details (including the choice of axioms) but it does not provide an alternative formal architecture for theories, which could replace the standard Hilbert-style architecture.

In particular, the "semantic" approach does not provide formal means for building and manipulating with models beyond the standard syntactic means. Bourbaki-style theories are *formalisable* rather than effectively formalised. They don't allow for an automated formal verification.

Marshall Stone 1961 in defence of Bourbaki-style mathematics (New Math)

"While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world."

Thus the task of developing formal architectures appropriate for representing scientific theories remains, in my view, wide open.

I approve on the view according to which a scientific theory is <u>not</u> a system of sentences (the "non-statement" view). However I'm going now to propose another version of this view supported by different formal means.

"On the Concept of Number" 1900

Here Hilbert extends his novel axiomatic method presented in his *Foundations of Geometry* of 1899 from geometry to arithmetic. For this purpose he introduces a distinction between what he calls the *genetic* and the axiomatic methods of introducing new theoretical concepts.

The genetic method is exemplified here by the well-known construals of real numbers from rational numbers due to Cauchy and Didekind: Cauchy Sequences and Dedekind Cuts.

In such cases a new mathematical concept (e.g. that of real number) is, in Hilbert's word, "produced" [erzeugt] by another concept (e.g. that of rational number).

"On the Concept of Number" 1900

Without trying to analyse this notion of "production" in logical terms Hilbert proposes replacing the traditional genetic theory of arithmetic by a formal axiomatic theory.

Comparing the two approaches Hilbert states without further ado that

"[D]espite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves the first rank."

"Foundations of Mathematics" v.1, 1934

"The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics."

"Foundations of Mathematics" v.1, 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory.

"Foundations of Mathematics" v.1, 1934

"Euclid's axiomatics was intended to be contentful and intuitive .... [I]ts axioms are not in existential form either: Euclid does not presuppose that points or lines constitute any fixed domain of individuals. [H]e does not state any existence axioms either, but only construction postulates."

Euclid's Elements: Postulates 1-3

1. to draw a straight-line from any point to any point.
2. to produce a finite straight-line continuously in a straight-line.
3. to draw a circle with any centre and radius.

Postulates 1-3 are elementary rules

postulate	input	output
P1	two points	straight line
P2	straight line	(longer) straight line
P3	straight line and its endpoint	circle

Euclid's Elements: Common Notions 1-3

Euclid's "axioms" 1-3 are inference rules:

$$\frac{a = c, b = c}{a = b} \tag{1}$$

$$\frac{a = b, c = d}{a + c = b + d} \tag{2}$$

$$\frac{a = b, c = d}{a - c = b - d} \tag{3}$$

Euclid's geometrical theory (El. books 1-4) in its original form is rule-based (Gentzen-style) rather than axiom-based (Hilbert-style).

Euclid's Postulates 1-3 are non-propositional rules: they input and output geometrical objects, not propositions.

Euclid's Common Notions aka Axioms are propositional rules, which are not logical by the modern standard since they apply only to propositions of a special sort, namely, to geometrical and arithmetical equalities.

Terminological observation: Rules and Axioms

A historical terminological point: Aristotle calls logical *rules* (e.g. that of "perfect" syllogism) by name of "axioms". The modern logical notion of axiom as self-evident (or conventially) true *proposition* is historically late.

Remarkably, the Zürich Axioms for investors proposed by Max Günter in 1985 are explicit rules.

Having this historical observation in mind I take a liberty to qualify Gentzen-style formal systems as *axiomatic* "in a broad sense".

Hilbert-style and Gentzen-style theories

Deduction Property (DP, aka Deduction Theorem for a given logical calculus):

$$\Gamma, F \vdash G \text{ if and only if } \Gamma \vdash F \rightarrow G \text{ for all } \Gamma,\ F \text{ and } G$$

allows one to see Hilbert-style and Gentzen-style axiomatic presentations of theories as equivalent; hence the talk of "style" of formal presentation.

Limits of Deduction Theorem(s)

However, such an equivalence does not hold universally since

- Certain physically-relevant logical calculi including Birkhoff-Neumann Quantum Logic do not enjoy DP;
- It is not clear how DP can be interpreted in *non-logical* Gentzen-style theories including scientific theories; it is reasonable to assume that such theories do not enjoy DP.

Logical and non-logical rules in the (formal) scientific reasoning

The rule-based mathematical and scientific reasoning typically involves non-logical rules such as Euclid's Postulates 1-3 or rules for conducting physical experiments that justify a given theory.

Hilbert's idea that in a final formal presentation of any theory such non-logical rules can be always dispensed with and fully replaced by logical rules (namely, MP and Substitution) applied to propositional axioms is a *very* strong epistemological assumption which, in my view, is not justified. Hilbert took this idea for granted without trying to justify it in a systematic way.

HoTT: Syntax

The syntactic carier of (the original version) HoTT is Martin-Löf's Constructive Type theory (with dependent types). MLTT is a Gentzen-style calculus that has no axioms.

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory."

HoTT: Syntax

Terms are points; (witnesses of) propositional equalities (identities) of terms $p : a =_T b$ are paths between these points.

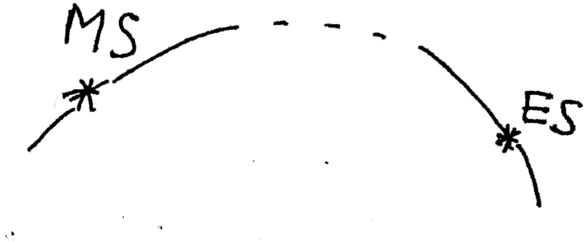

Рис.: The Morning Star is the Evening Star

Remark

The homotopical interpretation of MLTT belongs to its syntax rather than semantics; the involved spatial intuition is on par with one's capacity to distinguish between symbols of a given alphabet.

HoTT: Semantics

MLTT original semantics of judgement form $t : T$:

- t is an element of set T
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T

HoTT: Semantics

Homotopical level of a given type:

- (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $(x =_A y)$ are of h-level n.

HoTT: Semantics

Homotopical hierarchy of types:

(-2) single point *pt*;

(-1) the empty space $\emptyset$ and the point *pt* : truth values aka *classical* or "mere" propositions

(0) sets aka *intuitionistic* propositions aka theorems

(1) (flat) groupoids

(2) 2-groupoids

...

(n) n-groupoids

...

(ω) ω-groupoids

HoTT: Truncation

Each given type T of h-level $\geq$ (-1) can be *truncated* to propositional (-1)-type $\|T\|$ by collapsing its terms (if any) into one.

In case of equality $p, q : a =_T b$ this amounts to the forced formal identification of paths p, q.

The resulting proposition says that terms/points a, b of given type/space T are equal, i.e., that there exists a continuous path between them.

Thus rules of MLTT/HoTT apply to propositional and non-propositional, to wit geometrical, objects. Obtained higher-level constructions serve as proofs/witnesses of propositions obtained from their corresponding types by truncation. Compare the role of geometric constructions in Euclid's Elements.

HoTT effectively combines features of (Gentzen-style) axiomatic and *genetic* methods of theory-building without compromising the strict formal mode of presentation.

Revised notion of axiomatic theory

By a *theory* I understand a fragment of theoretical *knowledge* construed and represented as a system of theoretical *objects* (including propositional objects) supplied with *rules* for (human) manipulations with these objects.

By an *axiomatic theory* I understand a theory represented with a small number of distinguished *elementary objects* and *elementary rules*, which allow one to generate new objects from given objects and formulate new rules on the basis of given rules.

Some axiomatic theories

theory	elementary objects	elementary rules
Euclid	points and equalities	includes P1-3, CN1-3
Hilbert 1899	geometrical axioms	unspecified logical rules
Hilbert 1934	geometrical and logical axioms	MP and Sub
MLTT	atomic and base types	MLTT rules
HoTT/UF	points, base spaces, UA	MLTT rules

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Constructive View: key ideas:

- **Methods** are essential ingredients of scientific theories. This concerns not only the "context of discovery" but equally the "context of justification" of ready-made theories.
- Methods admit for a formal representation with Gentzen-style symbolic calculi. Such representations, generally, allow for computational implementation in form of programming code.
- Justification of scientific theories and single scientific judgements typically involves both logical and non-logical (in particular, empirical) methods. Non-logical methods have logical impact by producing evidences for judgements. Logical and non-logical methods are intertwined on the formal level (as in HoTT).

Theory interpretation/verification scheme

$$T \xrightarrow{m} M_T \xrightarrow{ct} C_T \xrightarrow{e} E_T$$

T : given formal theory;

M_T : background mathematical theory used for modelling T (e.g. Set theory, Category theory, etc.)

$m : T \rightarrow M_T$: mathematical model of T

C_T : setup for conducting computational and thought-experiments:

$ct : M_T \rightarrow C_T$ — thought/computational experiment, computational model of T.

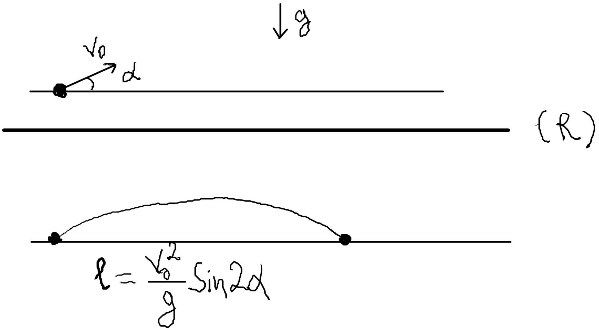
E_T : physical experimental setup

$e : C_T \rightarrow E_T$: physical experiment/observation that verifies T

Remark

Physical experiments and observations under the proposed reconstruction are fully theory-laden.

Example: a classical projectile



R is a derived rule in CM, not a basic rule.

Example: a classical projectile

$$R \xrightarrow{m} \mathbb{R}^2 \xrightarrow{t} V\mathbb{R}^2 \xrightarrow{e} PS$$

R — the above derived rule

$\mathbb{R}^2$: (theory of) Euclidean plane, standardly construed.

$V\mathbb{R}^2$ — visualised version of Euclidean plane geometry, possibly computer-assisted

PS — Earth's surface, slightly prepared; projectile device, instruments for measuring length and velocity.

Further Research

- To provide real examples of up-to-date physical theories constructively formalised. Candidate: GR and LIGO experiment.
- Open theoretical problem: Does the proposed notion of constructive axiomatic method restrict the received Hilbert-style axiomatic method, which allows one to stipulate theoretical entities without constructing them from their elementary parts?

THANK YOU!