Univalent Foundations and Applied Mathematics

in memoriam of Vladimir Voevodsky (1966-2017)

Andrei Rodin (andrei@philomatica.org)

Interdisciplinary Seminar on Algebraic Biology and System Theory, 15 December 2021

Voevodsky's Vision: Two Projects

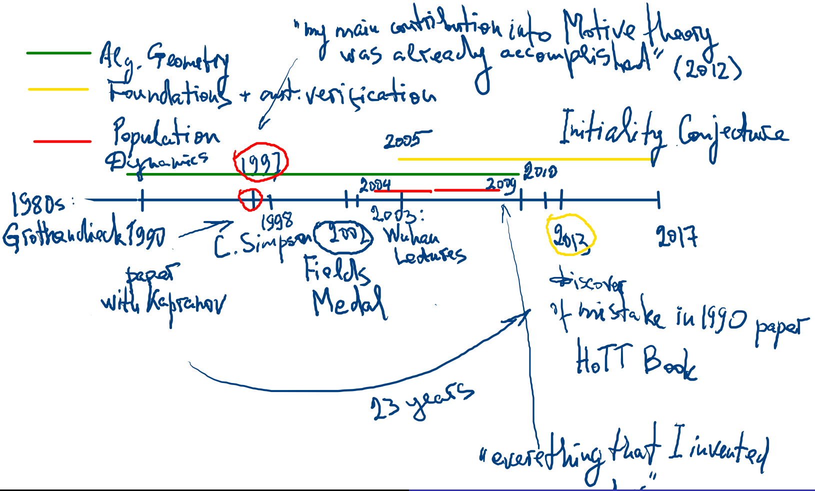
Pure and Applied Mathematics

Automated Proof Verification and Univalent Foundations

The Two Projects Merge

Vladimir Voevodsky (1966-2017)

Timeline


"my main contribution into Motive theory was already accomplished" (2012)
Alg. Geometry
Foundations + aut. verification
Population Dynamics
Initiality Conjecture
2005
1997
2010
2004
2009
1980s:
Grothendieck 1990
paper with Kapranov
1998
C. Simpson
2002
Fields Medal
2003: Wuhan Lectures
2013
2017
discover of mistake in 1990 paper
HoTT Book
23 years
"everything that I invented

What is most important for mathematics in the near future? (Wuhan, China, 2003

What is most important for mathematics in the near future? (Wuhan, China, 2003

- Connecting pure and applied mathematics;

What is most important for mathematics in the near future? (Wuhan, China, 2003

- Connecting pure and applied mathematics;
- Computerized version of Bourbaki.

From the interview to Roman Mikhailov, July 2012

Since Fall 1997 I realised that my main contribution into the Motive theory and Motive Cohomology was already accomplished. Since then I was very consciously and actively looking for [...] a theme to work on after I accomplish my obligations [...].

[C]onsidering tendencies of development of mathematics as a science I realised that we approach times when proving one more conjecture cannot change anything. [I realised] [t]hat mathematics is at the edge of crisis, more precisely, two crises.

From the interview to Roman Mikhailov, July 2012

The first crisis concerns the gap between the "pure" and applied mathematics. It is clear that sooner or later there will arise the question of why the society should pay money to people, who occupy themselves with things having no practical application.

The second crisis, which is less evident, concerns the fact that mathematics becomes very complex. As a consequence, once again, sooner or later mathematical papers will become too difficult for a detailed checking, and there will begin the process of accumulation of errors. Since mathematics is a very deep science in the sense that results of one particular paper usually depend on results of great many earlier papers, such an accumulation of errors is very dangerous for mathematics.

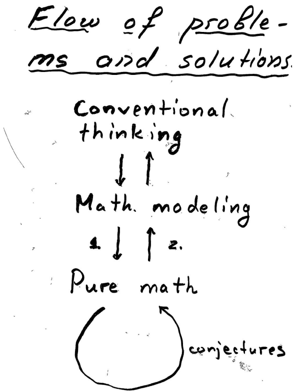
Voevodsky's Vision: Two Projects

Pure and Applied Mathematics

Automated Proof Verification and Univalent Foundations

The Two Projects Merge

Mathematics and the Outside World (Bangalore, India, 2003



The flow of problems down to the "mathematical modelling" level is filtered through the "computer modelling" level. As a result the "mathematical modelling" level, and as a consequence also the "pure mathematics" level, receive less problems than they used to receive before the rise of modern computer technologies.

This particularly affects today's abstract mathematics. Problems, which pass through the filter, are formulated in the old-style language of variables and analytic functions, while the language of today's abstract mathematics is the Set theory. Thus at least a part of problems received at the "pure mathematics" level pass through a double-translation, which further weakens the incoming flow of external problems into the pure mathematics

Double Translation of Problems

$$
\begin{tikzcd}
\text{Computer Model} \arrow[d] \arrow[dr, dashed] & \\
\text{Analytic language of MM} \arrow[r] & \text{Set-theoretic language of PM}
\end{tikzcd}
$$

Example: mathematical/computational models in Climate research

New Scheme of Relationships between the Computer Modelling and the Pure Mathematics

$$
\begin{tikzcd}
\text{Computer Model} \arrow[d] \\
\text{Set-theoretic Model} \arrow[d, "\textit{sometimes}"'] \\
\text{Analytic Model}
\end{tikzcd}
$$

In order to implement this new scheme we need to reformulate fundamental and applied scientific theories in the language of today's abstract mathematics, viz., in the set-theoretic language.

For this end we need to specify for each theory a notion of basic *unit* and then consider sets of such units.

Examples:

Science	Unit
Population Biology and Demography	Individuals
Financial Mathematics	Companies
Political Science	Voters
Particles Physics	Particles
Population Genetics	Genes
Future Theoretical Chemistry	Molecules

Cf. M. Stone 1961

in support of the New Math (Bourbaki-style) educational reform in school mathematics:

"While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world."

V.I. Arnold 1998

arguing **against** Bourbaki:

"Mathematics is a part of Physics. Physics is an experimental empirical science, a part of Natural Science. Mathematics is a part of Physics where experiments are cheap."

W.F. Lawvere 1970

[A] 'set theory' ... should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets, which do in fact develop along such parameters."

Lawvere's (generalised) 'set theory' is Topos theory

Voevodsky's Vision: Two Projects

Pure and Applied Mathematics

Automated Proof Verification and Univalent Foundations

The Two Projects Merge

History of one mathematical mistake

- 1990: joint paper in Russian with M. Kapranov motivated by the *Esquisse d'un Programme* by A. Grothendieck (1984): "∞-Groupoids as a Model for a Homotopy Category" first publsihed in УМН in Russian;
- 1998: Carlos Simpson (Laboratoire Dieudonné) claims a counter-example to the main theorem of Kapranov&Voevodsky 1990 (arXiv: 9810059). Voevodsky and Kapranov check their proofs but do not find mistakes in it. Simpson's paper is not published, the community suspects a mistake in the alleged counter-example
- 2013: Voevodsky finally finds a (uncorrectable) mistake in the original proof: the main theorem of the 1990 paper is a non-theorem!

Purpose of APV

The present situation when new alleged mathematical proofs can be checked by a few experts in the given field is hardly tolerable. The strong reliance on authority in mathematics blurs its objective character and rational nature. It makes research mathematics publicly indistinguishable from an esoteric sect led by a group of distinguished gurus and it makes it difficult to find applications of new mathematical results outside the Pure Mathematics.

APV solves the problem if

- Mathematical proofs are written in the form of computer code and the content of this code is conceptually transparent for all its competent users:
- The code is computationally effective and does not require computational resources .
- Epistemic transparency: users understand how the machine works and can reasonably evaluate the probability of error.

Voevodsky 2006

"Ideally, a paper submitted to a journal should contain text for human readers integrated with references to formalized proofs of all the results. Before being send to a referee the publisher runs all these proofs through a proof checker which verifies their validity. What remains for a referee is to check that the paper is interesting and that the formalizations of the statements correspond to their intended meaning."

History of the Idea:

- Descartes: Symbolic Algebra and Analytic Geometry;
- Leibniz: Geometrical Characteristics;
- Hilbert : Formal Axiomatic Method as "the basic instrument of all research"
- AUTOMATH (de Bruijn 1967), MIZAR (since 1973), HOL, Lego, Isabelle, Nuprl, Nqthm, AC2L, Elf, Plastic, Phox, PVS, IMPS, QED, ...

What has been achieved by 2000?:

Mostly *meta-mathematical* results such as Gödel's Incompleteness theorems. However important these results may be they have no direct relevance to the issue of formal proof checking.

Why formalization of mathematical reasoning did not become a common practice so far?

Because the existing principles and instances of formalization are NOT adequate!

Problems of set-theoretic formalization:

- Lack of invariance with respect to isomorphisms and higher equivalences (Benacerraf problem);
- The identification of proofs with formal deductions when a formal distinction between proof-supporting and not proof-supporting deductions is missing; (Prawitz, Martin-Löf);
- Formal deduction in ZFC, generally, is not algorithmic (implementability issues, cf. the case of MLTT).

Isomorphism-Invariance :

For any proposition P about object X and any isomorphism $X \cong X'$ there exists proposition P' about object X' such as P' is true if and only if P is true.

Breaking of II in the ZFC-coding:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

where

- $i \in \mathbb{N}$;
- $i \in \mathbb{Z}$

In ZFC whole numbers are encoded as ordered pairs of natural numbers. So in ZFC the two versions of the formula (for natural and whole numbers) are not logically equivalent.

Solution: a combination of the following

- Homotopy theory;
- theory of ∞-groupoids (Grothendieck);
- Martin-Löf Constructive Type theory (MLTT);
- prover COQ (after Thierry Coquand).

+ Univalence Axiom.

Main Features:

- Internal Logic
- Rules instead of Axioms (consider QTT);

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

Sets and Propositions Are the Same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [...] apart. Instead we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1983)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
(notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

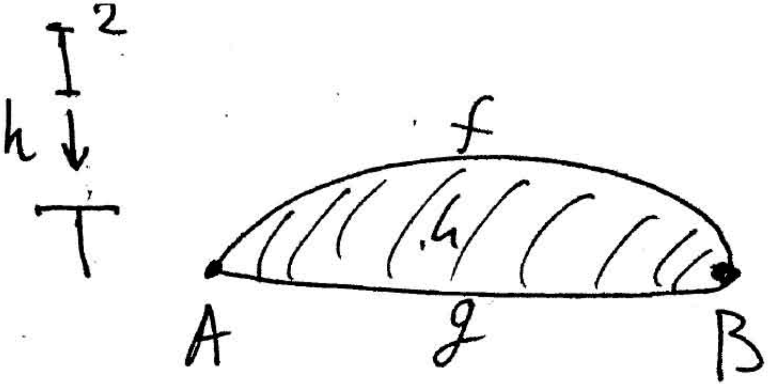
HoTT: the Idea

Types in MLTT are (informally!) modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of *n*-groupoids as higher homotopy groupoids of an appropriate topological space).

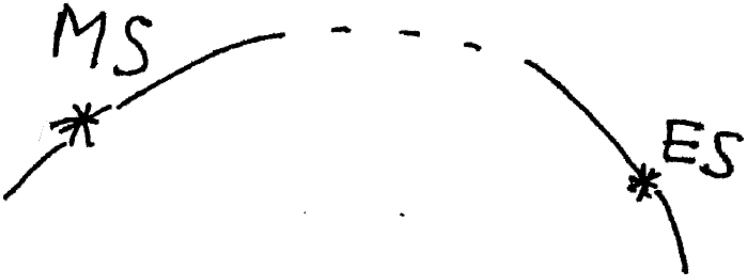
Homotopical interpretation of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

Homotopy



Morning Star and Evening Star are the same planet



Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).

Homotopy Levels

Definition

We say that S is a space of h-level $n+1$ if for all its points x, y path spaces $x \; =_S \; y$ are of h-level n.

Cummulative Hierarchy of Homotopy Types

- -2-type: single point pt;
- -1-type: the empty space $\emptyset$ and the point pt: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Univalence Axiom

$$(A =_{TYPE} B) \simeq (A \simeq B)$$

In words: equivalnce of types is equivalent to their equality.

For PROPs: $(p = q) \leftrightarrow (p \leftrightarrow q)$ (propositional extensionality)

For SETs: Propositions on isomorphic sets are logically equivalent (isomorphism-invariance)

Univalence implies *functional extensionality*: if for all $x \quad X$ one has $fx =_Y gx$ then $f =_{X \to Y} g$ (the property holds at all h-levels).

Voevodsky's Vision: Two Projects

Pure and Applied Mathematics

Automated Proof Verification and Univalent Foundations

The Two Projects Merge

Voevodsky 2012 on Pure and Applied Mathematics

"Concerning pure and applied mathematics, I have the following picture. Pure mathematics is working with models of high abstraction and low complexity (mathematicians like to call this low complexity elegance). Applied mathematics is working with more concrete models but on the higher complexity level (many equations, unknowns, etc).

Interesting application of the modern pure mathematics are most likely in the area of high abstraction and high complexity. This area is practically inaccessible today, mostly due to the limitations of the human brain [. . .]".

Voevodsky 2012 on the potential role of UF in Applied Mathematics

"When we will learn how to use computers for working with abstract mathematical objects this problem will be no longer important and interesting applications of ideas of today's abstract mathematics will be found. "

"That is why I think that my present work on computer languages that allow one to work with such objects, will be also helpful for application of ideas of today's pure mathematics in applied problems. "

(The context and timing make it clear that taking about his "work on computer languages" Voevodsky refers here to his research in the Univalent Foundations and its computational implementation.)

Voevodsky 2012 on Data-Driven Science

"Science should collect and comprehend a new knowledge. The collection part is very important. There is a view according to which all important observations are already done, the general world image is clear, so it remains only to arrange this knowledge and pack it into a compact and elegant theory. "

"This view is wrong. It is not only wrong but also supports a very negative tendency to ignore everything that doesn't fit a ready-made theory or hypothesis. This is one of the most important problems of today's science."

Is UF application-friendly? Some reasons for hope:

Is UF application-friendly? Some reasons for hope:

- Computational implementation;

Is UF application-friendly? Some reasons for hope:

- Computational implementation;
- Spatial (to wit, homotopical) intuition as a join between experience/measurement and mathematical formalism;

Is UF application-friendly? Some reasons for hope:

- Computational implementation;
- Spatial (to wit, homotopical) intuition as a join between experience/measurement and mathematical formalism;
- Topological Data Analysis as a novel mathematical instrument for theory-building in empirical sciences (including Brain Science, Biology, Geosciences, etc.)

СПАСИБО!