Kolmogorov and Voevodsky on the Scope of Logic

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Plan:

1. Problems and Theorems
2. Kolmogorov's Calculus of Problems
3. Galois connection between problems and propositions
4. Homotopy Type theory
5. Conclusions

Problems and Theorems

For isosceles triangles, the angles at the base are equal to one another. [. . .] (Which is) the very thing it was required **to show**. (El. 1.5)

To construct an equilateral triangle on a given finite straight-line. [. . .] (Which is) the very thing it was required **to do**. (El. 1.1)

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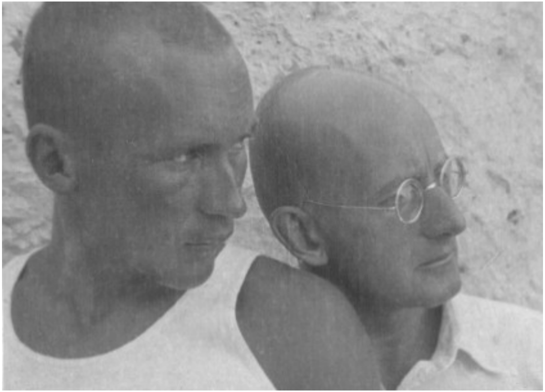
In the deductive structure of the 'Elements' problems and theorems are intertwined. Postulates and Axioms are applied both in problems and in theorems.

Kolmogorov and Alexandrov in Germany in 1931

Kolmogorov and Alexandrov



Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Calculus of Problems 1932:

(1) Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [...] If we can reduce the solution of problem b to the solution of problem a, and the solution of problem c to the solution of problem c to the solution of problem b, then the solution of c can also be reduced to the solution of a.

Calculus of Problems 1932:

(2) The following remarkable fact holds: the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting" [reference to *Die formalen Regeln der intuitionistischen Logik*, 1930, in two parts]

Commentary of 1985:

Paper "On the interpretation of intuitionistic logic" was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types — propositions and problems.

Cf. the intuitionistic notion of proposition: Heyting 1934

Each mathematical proposition [. . .] is an intention towards a mathematical construction, which should satisfy certain conditions.

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Corollary: propositions and problems are the same.

Cf. the intuitionistic notion of proposition: D. Prawitz paraphrased by P. Martin-Löf 1984

A proposition is defined by laying down what counts as a proof of the proposition. . . . A proposition is true if it has a proof, that is, if a proof of it can be given.

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When we hold a proposition to be true we make a *judgement*.

S. Melikhov 2022: unified calculus of problems and propositions

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Galois connection: $\vdash ?\alpha \rightarrow p$ if and only if $\vdash \alpha \rightarrow !p$.

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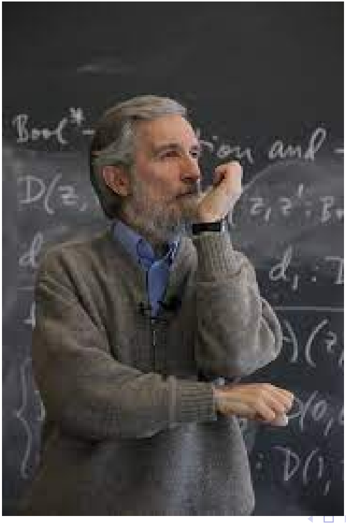
- $\nabla\nabla\alpha \to \nabla\alpha$;
- $(\alpha \to \beta) \to (\nabla\alpha \to \nabla\beta)$.

Melikhov on the unified calculus of problems and propositions

https://arxiv.org/abs/1312.2575
https://arxiv.org/abs/1504.03379

Per Martin-Löf

1964-65: work under the supervision of A.N. Kolmogorov



Alternative explanations / interpretations of judgements in MLTT

In MLTT(1984) there are four different forms of judgement; here is how Martin-Löf explains the judgement form $a\,A$ where A is a type and a is a term of this type; in fact, he proposes several apparently different readings of this formula and argues that they are essentially the same :

1. a is an element of set A
2. a is a proof (witness, evidence) of proposition A
3. a is a method of fulfilling (realising) the intention (expectation) A
4. a is a method of solving the problem (doing the task) A

Propositions and Sets

'If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . .] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead we simply identify them, that is, treat them as one and the same notion."

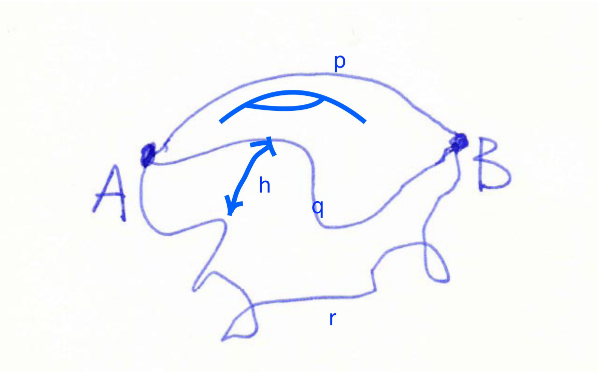
Higher Identity Types

- $p, q : P =_T Q$
- $p', q' : p =_{P =_T Q} q$
- $\ldots$

Vladimir Voevodsky

Voevodsky circa 2010: types as (fundamental groupoids of) homotopy spaces

$A, B : T; \mathrm{T} : \mathrm{U}$
$p, q, r : A =_T B$
$h : (q =_{A=B} r)$
$(p =_{A=B} q) =_U \emptyset$



Homotopical hierarchy of types for judgement *a A*

Definition: S is a space of h-level $n+1$ if for all its points x, y path spaces $x \;=_S\; y$ are of h-level n. where h-level is read as as the homotopy level.

- h-level (-2): single point pt;
- h-level (-1): the empty space $\emptyset$ and the point pt: truth-values aka (mere) propositions
- h-level 0: sets (discrete point spaces)
- h-level 1: flat path groupoids : no non-contractibe surfaces
- h-level 2: 2-groupoids : paths and surfaces but no non-contractible volumes
-
- h-level n: n-groupoids
- ...
- h-level ω: ω-groupoids

A top-down cumulative character of the homotopical hierarchy

Every k-type is a n-type for all $n > k$.

Every proposition is a set (either the empty set or a singleton), every set is a trivial flat groupoid (without paths save reflections), every flat groupoid is a trivial 2-groupoid (without path homotopies), etc.

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$T^k \to T^m, m < k$

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Propositional truncation: $T^k \to T^{(-1)}$

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According to this interpretation a judgement of form $a : A$ is not, generally, analysed into a proposition A and its proof a. A may turn out to be a higher-order type and a a higher-order construction, which makes true the underlying proposition $A^{(-1)}$.

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How the two developments are related remains an interesting open question.

THANKS!