Kolmogorov's Calculus of Problems and Its Posterity

Andrei Rodin

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Plan:

Problems and Theorems in Euclid

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example of problem: El. 1.5:
To construct an equilateral triangle on a given finite straight-line. [...] (Which is) the very thing it was required **to do**.

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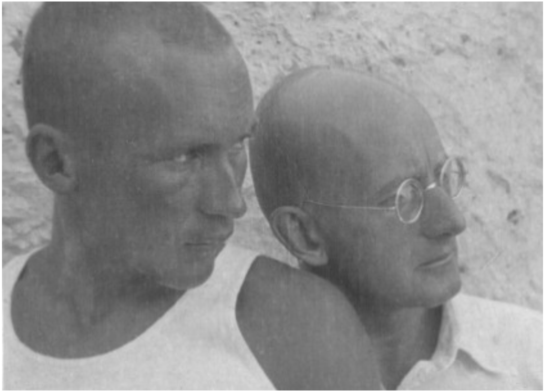
Remarks:
In the deductive structure of the 'Elements' problems and theorems are *intertwined*. Postulates and Axioms are applied both in problems and in theorems. Earlier solved problems are used in proofs of new theorems and earlier proved theorems are used in solutions of new problems.

A faithful formal reconstruction of the deductive structure of Euclid's geometry remains an open problem. Problems in Euclid's sense of the word remain pertinent in today's mathematics.

Kolmogorov and Alexandrov in Germany in 1931

Kolmogorov and Alexandrov



Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Calculus of Problems 1932:

Kolmogorov: Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [. . .] If we can reduce the solution of problem b to the solution of problem a, and the solution of problem c to the solution of problem c to the solution of problem b, then the solution of c can also be reduced to the solution of a.

Calculus of Problems 1932:

Kolmogorov: The following remarkable fact holds: the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting" [reference to *Die formalen Regeln der intuitionistischen Logik*, 1930a, in two parts]

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Remark: $P \vee \neg P$ where P is a problem reads:

There exists a *general* (constructive) method of solving the following problem: given *any* problem P either solve P or prove that P is unsolvable.

Calculus of Problems 1932:

Kolmogorov: It will be shown that intuitionistic logic *should* be replaced by the calculus of problems, since its objects are, in fact, problems rather than theoretical propositions. [emphasis mine]

Kolmogorov's commentary of 1985:

Kolmogorov: Paper "On the interpretation of intuitionistic logic" was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types — propositions and problems.

Uspenskii&Plisko 1991 on Kolmogorov 1932

In Kolmogorov 1925 an embedding operation [= the double negation translation of CL formulas into IL formulas, cf. Glievenko 1929 and Gödel 1933] is constructed which makes it possible to give an intuitionistic interpretation to the major part of classical mathematics, while, in a sense, the paper Kolmogoroff 1932 is devoted to the solution of the inverse problem of interpreting intuitionistic logic within the framework of ordinary mathematical notions, irrespective of the philosophical and methodological principles of intuitionism.

Kolmogorov's interpretation of IL in the nutshell :

The "intuitionistic logic" (that is, Heyting's 1930a propositional calculus) needs not and should not be interpreted *intuitionistically* (that is, according to philosophical principles of Mathematical Intuitionism).

Kolmogorov's Preface to Russian edition of Heyting 1934 appeared in 1936

Kolmogorov: We cannot agree with the intuitionists when they claim that mathematical objects are products of the constructive activity of our spirit. For us, mathematical objects are abstractions from existing forms of reality, which is independent from our spirit. We know that the constructive solutions of problems are as much important in mathematics as the pure proofs of theoretical sentences. This constructive aspect of mathematics does not conceal for us its other and more fundamental aspect, namely, its epistemic aspect. But the laws of mathematical construction discovered by Brouwer and systematised by Heyting under the appearance of new intuitionistic logic, so understood, preserve for us their fundamental significance.

Heyting 1930b

Heyting: A proposition p like, for example, "Euler constant $[C]$ is rational" expresses a *problem* [(un problème)], or, better yet, a certain expectation [(une certaine attente)] (that of finding two integers a and b such that $C = \frac{a}{b}$), which can be fulfilled (réalisé) or disappointed (déçue).

Heyting 1931

Heyting: A mathematical proposition [(Aussage)] expresses a certain expectation [(Erwartung)]. For example, the proposition, "Euler constant C is rational" expresses [(bedeutet)] the expectation that we could find two integers a and b such that $C = \frac{a}{b}$. Perhaps, the word "intention" [(Intention)], coined by the phenomenologists, expresses even better what is meant here.

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Corollary: propositions and problems are the same.

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Remark: Unlike Heyting Kolmogorov doesn't use the expression "intuitionistic mathematics". Unlike Heyting Kolmogorov does not attempt to develop a special kind of logic and/or mathematics, which could either replace or co-exist with the "usual" (Classical) logic and mathematics. But Kolmogorov shares with Heyting and other intuitionists the view that LEM is not applied universally. Namely, it doesn't apply to mathematical *problems*. It is hard to say whether Kolmogorov holds the view that LEM applies to all *propositions* (see Melikhov below). Apparently he wants (but does not propose in 1932 or in later works) a refined notion of proposition that satisfies this condition. In a letter to Heyting Kolmogorov remarks that Π^0_1 formulas are subjects to LEM but to identify propositions with this class of formulas would be two restrictive.

Heyting 1958: the difference is ignored

Heyting: The older interpretations by Kolmogoroff (as a calculus of problems) and Heyting (as a calculus of intended construction) were substantially equivalent.

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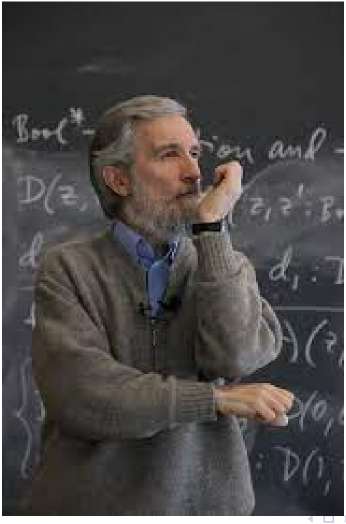
Troelstra 1977 ; 1990: Kreisel $\rightarrow$ Kolmogorov

Per Martin-Löf

1964-65: work under the supervision of A.N. Kolmogorov



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4. a is a method of solving the problem (doing the task) A

PML 1984: extended BHK: sets and propositions are the same

Martin-Löf: If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [...] apart. Instead we simply identify them, that is, treat them as one and the same notion.

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Corollary: propositions and sets and problems and expectations (to solve a problem) and intentions (towards a construction) are all the same.

Historical Claim:

The notion of BHK interpretation is justified as a theoretical proposal (whatever is its name). But the historical thesis according to which Kolmogorov's 1932 intended interpretation of IL is essentially the same as Heyting's 1934 interpretation (modulo the dispensable semantic differences between terms "problem" and "intention" and "expectation") is false.

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Yet, this historical error (with some variations) is systematically made by historians of the intuitionistic school (van Dalen, Troelstra, Sundholm, van Atten) who follow Heyting's 1958 remark on Kolmogorov without taking seriously Heyting 1934. As historians of the intuitionistic logic and mathematics and as adherents of Mathematical Intuitionism themselves these scholars tend to ignore the fact that Kolmogorov rejects the basic epistemological principles of Mathematical Intuitionism and pursues an alternative foundational agenda.

van Dalen 1979:

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Commentary: [1] is correct. [2] is based on the assumption that (i) Heyting's view on foundations is correct but Kolmogorov's view on foundations is wrong or (ii) (more plausibly) that Kolmogorov's interpretation has no foundational ambition at all. But (ii) is evidently false, see Kolmogorov 1929.

Kolmogorov 1929

Kolmogorov: The most common issue from this difficulty [of non-constructive mathematical existence] among *mathematicians that avoid philosophy* is limitation of the domain of "existence". [. . .].

This position — though the most placid one — suffers from unprincipledness, which is expressed most evidently in the fact that bounds of what each mathematician is "ready to admit" depend on his personal interests.

After 1932..

Kolmogorov did not publish logical works after 1932.

In 1979, however, he became the Chair of Logic in the Mathematical Department of MSU...

2 operators

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Monotonicity of ? and !:
$\vdash \alpha \rightarrow \beta$ implies $\vdash ?\alpha \rightarrow ?\beta$
$\vdash p \rightarrow q$ implies $\vdash !p \rightarrow !q$

Galois connection

Let $< A, \leq >$, $< B, \leq >$ be posets and $f\ A \to B$, $g\ B \to A$ be two monotone (order preserving) functions. Monotone Galois connection is a pair (f, g) such that for all $a \in A$ and all $b \in B$, $f(a) \leq b$ if and only if $f(b) \leq a$

Lemma: If (f, g) is a Galois connection then one of these functions uniquely determines the other.

Remark: $g \circ f\ A \to A$ is a closure operator while $f \circ g\ B \to B$ is a kernel (interior) operator.

Galois connection in LPP

$\vdash ?\alpha \to p$ if and only if $\vdash \alpha \to !p$.

Closure and interior operators:

interior: $\Box p := ?!p$;
closure: $\nabla\alpha := !?\alpha$

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- $\nabla\nabla\alpha \to \nabla\alpha$ (left idempotency: second clause in Kreisel);
- $(\alpha \to \beta) \to (\nabla\alpha \to \nabla\beta)$ (distribution).

Melikhov on the unified calculus of problems and propositions

https://arxiv.org/abs/1312.2575
https://arxiv.org/abs/1504.03379

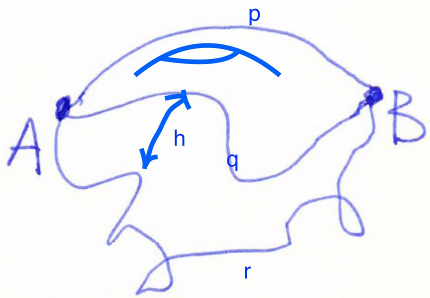
Higher Identity Types

- $p, q : P =_T Q$
- $p', q' : p =_{P =_T Q} q$
- $\dots$

Vladimir Voevodsky

Voevodsky circa 2010: types as (fundamental groupoids of) homotopy spaces

$A, B : T; \mathrm{T} : \mathrm{U}$
$p, q, r : A =_T B$
$h : (q =_{A=B} r)$
$(p =_{A=B} q) =_U \emptyset$



Homotopical hierarchy of types for judgement $a\ A$

Definition: S is a space of h-level $n+1$ if for all its points x, y path spaces $x \ =_S \ y$ are of h-level n. where h-level is read as as the homotopy level.

- h-level (-2): single point pt;
- h-level (-1): the empty space $\emptyset$ and the point pt: truth-values aka (mere) propositions
- h-level 0: sets (discrete point spaces)
- h-level 1: flat path groupoids : no non-contractibe surfaces
- h-level 2: 2-groupoids : paths and surfaces but no non-contractible volumes
-
- h-level n: n-groupoids
- ...
- h-level ω: ω-groupoids

A top-down cumulative character of the homotopical hierarchy

Every k-type is a n-type for all $n > k$.

Every proposition is a set (either the empty set or a singleton), every set is a trivial flat groupoid (without paths save reflections), every flat groupoid is a trivial 2-groupoid (without path homotopies), etc.

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$T^k \to T^m, m < k$

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Propositional truncation: $T^k \to T^{(-1)}$

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According to this interpretation a judgement of form $a : A$ is not, generally, analysed into a proposition A and its proof a. A may turn out to be a higher-order type and a a higher-order construction, which makes true the underlying proposition $A^{(-1)}$.

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How the two developments are related (if they are) is an interesting open question.

THANKS!