Proofs and Solutions, according to Kolmogorov

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Workshop Proofs and Styles of Reasoning across History and Cultures

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Plan:

1. BHK-interpretation
2. Intuitionistic appropriation of Kolmogorov's Calculus of Problems
3. Problems and Propositions
4. Legacy of CP beyond the BHK-interpretation

Proof-intertretation of IL aka BHK-interpretation (for Brouwer, Heyting and Kolmogorov, see Troelstra& van Dalen 1988)

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- A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B;
- A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B;
- Absurdity $\bot$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

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- 1990: A.S. Troelstra in "The Early History of Intuitionistic Logic": a historical dimension.

Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Popular perception of Kolmogorov's Calculus of Problems

https://ncatlab.org/nlab/show/BHK+interpretation

> *The idea of the [BHK] interpretation is clearly expressed in Kolmogorov (1932, p. 59), though rather briefly and in unusual terminology: Instead of propositions, Kolmogorov speaks of Aufgaben (Deutsch for "tasks", but here in the sense used in math classes where it means "exercises" or "mathematical problems") [. . .].*

Intuitionistic Mathematics

Heyting systematically refers in his publications to the concept of "intuitionistic mathematics" [Intuitionistische Mathematik in German, IM for short] since 1930 on. He describes IM as a self-sustained corpus of mathematical knowledge with its proper standard of rigour and admissible forms of mathematical reasoning.

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Heyting's *Intuitionism* (1956) shows how his project of developing IM evolves over the years. In addition to foundations of intuitionistic mathematics this monograph includes chapters on (the constructive counterparts of) Linear Algebra and Geometry, Measure theory, elements of Functional Analysis (Hilbert spaces) and, last but not least, Logic (which according to the intuitionistic epistemic standard does not belong to the foundations).

Troelstra and van Dalen

Anne Sjerp Troelstra and Dirk van Dalen were Arend Heyting's students who further developed the same project under his supervision. Troelstra's Ph.D. thesis (1966) is on the "Intuitionistic General Topolgy" ; van Dalen's thesis (1963) is on the "Intuitionistic Plane Projective Geometry".

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Troelstra&vanDalen's work in Logic including their notion of BHK-interpretation (1988) is a part of this larger project.

Kolmogorov's Motivation

By contrast, Andrei Nikolaevitch Kolmogorov, disagreed with the intuitionistic philosophy, and did not accept the very notion of "intuitionistic mathematics".

> *We cannot agree with the intuitionists when they claim that mathematical objects are products of the constructive activity of our spirit. For us, mathematical objects are abstractions from existing forms of reality, which is independent from our spirit. We know that the constructive solutions of problems are as much important in mathematics as the pure proofs of theoretical sentences. This constructive aspect of mathematics does not conceal for us its other and more fundamental aspect, namely, its epistemic aspect. But the laws of mathematical construction discovered by Brouwer and systematized by Heyting under the appearance of new intuitionistic logic, so understood, preserve for us their fundamental significance (1936).*

Calculus of Problems 1932:

Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [...] If we can reduce the solution of problem b to the solution of problem a, and the solution of problem c to the solution of problem b, then the solution of c can also be reduced to the solution of a.

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The following remarkable fact holds: the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting.

Kolmogorov's commentary of 1985:

Paper "On the interpretation of intuitionistic logic" was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types — propositions and problems.

Heyting's acknowlegement

In 1934 Heyting acknowledges the differences between his intended interpretation of IL and Kolmogorov's interpretation (my translation from German):

> *Kolmogorov developed an akin [verwandten] idea which, however, goes beyond the former [Heyting's] idea rsince it provides Heyting's calculus with a meaning that does not depend on the intuitionistic assumptions [intuitionistischen Voraussetzungen].*

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In 1958, however, Heyting described Kolmgorov's interpretation of 1932 and his own contemporary interpretation as essentially the same:

> *The older interpretations by Kolmogoroff (as a calculus of problems) and Heyting (as a calculus of intended construction) were substantially equivalent.*

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My study of the historical sources shows that Heyting's 1934 view on Kolmogorov's contribution was accurate but his later 1958 view was not accurate.

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The notion of BHK-interpretation of IL is fully justified theoretically as any fruitful combination of ideas whatever are their sources. But one needs to bear in mind that

- BHK-interpretation does not provide a faithful representation of Kolmogorov's intended interpretation of IL; it provides instead an "intuitionistic projection" of Kolmogorov's interpretation;
- BHK-interpretation cannot be used as a sufficient theoretical framework for a historical study of Kolmogorov's logical works; such a study requires taking into consideration a wider theoretical context that involves relevant developments outside the intuitionistic (constructive) logic and mathematics.

Heyting 1930b

> *A proposition p like, for example, "Euler constant [C] is rational" expresses a problem [(un problème)], or, better yet, a certain expectation [(une certaine attente)] (that of finding two integers a and b such that $C = \frac{a}{b}$), which can be fulfilled (réalisé) or disappointed (déçue).*

Remark that Heyting explains here propositions in terms of *problems* before Kolmogorov's 1932 paper appears in press. Given the lack of historical evidence to the contrary, one can assume that Heyting does this independently from Kolmogorov.

Heyting 1931

A mathematical proposition [(Aussage)] expresses a certain expectation [(Erwartung)]. For example, the proposition, "Euler constant C is rational" expresses [(bedeutet)] the expectation that we could find two integers a and b such that $C = \frac{a}{b}$. Perhaps, the word "intention" [(Intention)], coined by the phenomenologists, expresses even better what is meant here.

Heyting 1934: the mature view

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Corollary: propositions and problems are the same.

Kolmogorov to Heyting, October 12, 1931

Every "proposition" [p] in your conception is, in my view, of one of the following two kinds:

(α) p expresses the hope that, in some circumstances or other, some experiment will always give a definite result (for example, that the attempt to decompose any [given] even number n into the sum of two primes [in Goldbach Conjecture] gives a positive result [...]. Naturally, every "experiment" must be realizable by means of a finite number of determined operations.

(β) p expresses the intention to find a certain construction.

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I prefer to reserve the name of proposition [Aussage] only for propositions of the form (α) and to call "propositions" of the form (β) simply problems [Aufgaben].

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With the proposition p are associated the problems $\neg p$ (to reduce p to a contradiction) and $+p$ (to prove p).

Kolmogorov to Heyting?

In a following undated letter Kolmogorov rejects pseudo-constructive procedures such as mathematical "experiments that always give a definite result", which opens him a way to more sharply between *proposition* p and the associated *problem* $+p$ (to prove p) without requiring that the wanted prove is constructive.

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Heyting, in his turn, in 1934 fuses problems and propositions more systematically by abandoning the $+$ operator that he used earlier. By Goran Sundholm's word here Heyting "commits himself" [to his mature version of intuitionistic semantics]. Heyting's semantics of intuitionistic propositions is the core of the BHK semantics.

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Thus Kolmogorov and Heyting find different and even opposite ways out of the same theoretical difficulty.

Kolmogorov 1929

The law of excluded middle according to Brouwer could not be applied only to a certain kind of judgements, in which a theoretical statement is closely connected with construction of the object of the statement. Therefore, we may assume that Brouwer's ideas do not contradict the traditional logic, which has never before dealt with such judgements.

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Kolmogorov's take on the logical difficulty concerning LEM pointed to by Brouwer is to distinguish between problems and problems more systematically and thus avoid mistaking problems for propositions, whatever is their linguistic form. This task is not accomplished in Kolmogorov's 1932 paper that treats the logic problems but does not formally connect it to the logic of propositions.

Are Kolmogorov's logical ideas beyond the BHK-interpretation theoretically valuable?

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- Computability Logic by Giorgi Japaridze (since early 2000s)
- Combined Logic of Problems and Propositions QHC by Sergei Melikhov (since 2015)

Cultural Remark

All the aforementioned researchers belong to Russian school of mathematical logic and are influenced by Kolmogorov directly.

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In 2016 Medvedev's and Muchnik's ideas have been taken up and further developed by S.S. Basu and S.G. Simpson (with their novel concept of Muchnik Topos) quite independently of Russian connections.

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Monotonicity of ? and !:
$\vdash \alpha \rightarrow \beta$ implies $\vdash ?\alpha \rightarrow ?\beta$
$\vdash p \rightarrow q$ implies $\vdash !p \rightarrow !q$

Galois connection

Let $< A, \leq >$, $< B, \leq >$ be posets and $f\ A \rightarrow B$, $g\ B \rightarrow A$ be two monotone (order preserving) functions. Monotone Galois connection is a pair (f, g) such that for all $a \in A$ and all $b \in B$, $f(a) \leq b$ if and only if $f(b) \leq a$

Lemma: If (f, g) is a Galois connection then one of these functions uniquely determines the other.

Remark: $g \circ f\ A \rightarrow A$ is a closure operator while $f \circ g\ B \rightarrow B$ is a kernel (interior) operator.

Galois connection in QHC

$\vdash ?\alpha \to p$ if and only if $\vdash \alpha \to !p$.

Closure and interior operators:

interior: $\Box p := ?!p$;
closure: $\nabla\alpha := !?\alpha$

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- $\nabla\nabla\alpha \to \nabla\alpha$ (left idempotency: second clause in Kreisel);
- $(\alpha \to \beta) \to (\nabla\alpha \to \nabla\beta)$ (distribution).

Preliminaries to the Univalent Mathematics: MLTT

In MLTT(1984) there are four different forms of judgement; here is how Martin-Löf explains the judgement form $a : A$ where A is a type and a is a term of this type (extended BHK-interpretation). PML proposes the following explanations / interpretations of judgements, which are mutually complimentary (that is, they differ in their linguistic form but not conceptually):

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3. a is a method of fulfilling (realising) the intention (expectation) A;
4. a is a method of solving the problem (doing the task) A.

PML 1984: sets and propositions are the same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [...] apart. Instead we simply identify them, that is, treat them as one and the same notion.

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- mutatis mutandis for higher homotopies.

Homotopical hierarchy of types for judgement $a\ A$

Definition: S is a space of h-level $n+1$ if for all its points x, y path spaces $x =_S y$ are of h-level n.

- h-level (-2): single point pt;
- h-level (-1): the empty space $\emptyset$ and the point pt: truth-values aka (mere) propositions
- h-level 0: sets (discrete point spaces)
- h-level 1: flat path groupoids : no non-contractibe surfaces
- h-level 2: 2-groupoids : paths and surfaces but no non-contractible volumes
-
- h-level n: n-groupoids
- ...
- h-level ω: ω-groupoids

A top-down cumulative character of the homotopical hierarchy

Every k-type is a n-type for all $n > k$.

Every proposition is a set (either the empty set or a singleton), every set is a trivial flat groupoid (without paths save reflections), every flat groupoid is a trivial 2-groupoid (without path homotopies), etc.

Truncation (m<k)

$T^k \to T^m, m < k$

A "mere" proposition P, if not empty, collapses all proofs of P into a single truth-value *true*.

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Propositional truncation: $T^k \to T^{(-1)}$

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What is a judgement? (once again)

The homotopical hierarchy of types is at odds with Martin-Löf's intended interpretation according to which propositions, sets and higher-order constructions are essentially the same. In HoTT propositions and sets are types of different homotopical levels: every proposition is a set (either empty or singleton) but not every set is a proposition.

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HoTT supports Kolmogorov's view on problems and propositions (without being motivated by Kolmogorov's work): propositional types are types of a special kind; higher types are naturally interpreted as problems (and their terms as higher-order constructions that solve those problems).

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Notice that this time Kolmogorov's distinction between general problems and propositions is interpreted in a purely constructive setting. Here proposition p and problem "to prove p" are the same (as in BHK) but solution of a higher-order problem does not reduce to proving a proposition.

Open Problem

Work out theoretical (syntactic and semantic) connections between Epistemic (modal) Logic and UF/HoTT.

Andrei Rodin, Kolmogorov's Calculus of Problems and Its Legacy
https://arxiv.org/abs/2307.09202

THANKS!