5.1: Counting

Most of us think that counting is as easy as 1, 2, 3... When counting objects, one needs to be careful to not count an object more than once or miss an object. In this section, we will explore some ideas behind counting.

The Multiplication Principle

If a process can be broken down into two steps, performed in order, with m ways of completing the first step and n ways of completing the second step after the first step is completed, then there are (m)(n) ways of completing the process.

Example 5.1.1:

Suppose that pizza can be ordered in 3 sizes, 2 crust choices, 4 choices of toppings, and 2 choices of cheese toppings. How many different ways can a pizza be ordered?

Solution

To determine the number of possibilities, we will use the multiplication principle. Let $S =$ pizza size, $C =$ crust choice, $T =$ topping choice, and $Ch =$ cheese choice.

Since we need to choose one choice from each category, we can write that we need to choose size, and crust, and topping, and cheese.

Let's use the multiplication principle:

$$\begin{aligned} \text{Ways} &= (S)(C)(T)(Ch) \\ &= (3)(2)(4)(2) \\ &= 48 \end{aligned}$$

What if there was only one choice for cheese? How would this affect the calculation?

Example 5.1.2:

Count the number of possible outcomes when:

1. A coin tossed four times.
2. A standard die is rolled five times.

Permutation

Definition: Permutation

A permutation is an ordered arrangement of objects.

The number of permutations of n distinct objects, taken all together, is $n!$, where

$$n! = n(n-1)(n-2).\dots.1 \tag{5.1.1}$$

.

Note that $0! = 1$.

Example 5.1.3

Miss James wants to seat 30 of her students in a row for a class picture. How many different seating arrangements are there? 17 of Miss James' students are girls and 13 are boys. In how many different ways can she seat 17 girls together on the left, then the 13 boys together on the right?

Solution

Let's start with the girls. There are 17 of them, and so, when seating the first girl in the row, there are 17 choices. The next spot will have 16 choices left, then 15, and so on. Thus, the number of choices for seating the girls can be written $17!$.

For the boys, by the same reasoning, there are $13!$ ways to seat them on the right.

Now let's apply the multiplication principle: we need to seat the girls and the boys at the same time. For each permutation we might pick for the girls, we need to apply each different case for the boys as a distinct possibility. So, our result is $(17!)(13!)$. This means there are $2.215 \cdot 10^{24}$ different ways to seat these students with girls on the left and boys on the right!

Permutations

The number of permutations of r objects picked from n objects, where $0 \leq r \leq n$, is

$$_nP_r = \frac{n!}{(n-r)!}. \tag{5.1.2}$$

When reading this out loud, we say "n Pick r" - when we pick something, like a team for sports or favorite desserts, the order matters.

Example 5.1.4:

Using the digits 1,3,5,7, and 9, with no repetitions of digits, how many three–digit numbers can be made?

Solution

We have $n = 5$ objects, and we want to pick $r = 3$ of them. So via Equation 5.1.2:

$$\begin{aligned} _nP_r &= \frac{5!}{(5-3)!} \\ &= \frac{5!}{2!} \\ &= \frac{(5)(4)(3)(2)(1)}{(2)(1)} \\ &= (5)(4)(3) \\ &= 60 \end{aligned}$$

Combination

The following is defined already in 3.3 Finite Difference Calculus.

Combinations

The number of combinations of r objects chosen from n objects, where $0 \leq r \leq n$, is

$$_nC_r = \frac{n!}{(n-r)!r!} \tag{5.1.3}$$

$_nC_r$ is also denoted as $\binom{n}{r}$. When reading this out loud, we say "n Choose r" - when we choose objects, like candies out of a bag or clothes from a closet, the order doesn't matter.

Example 5.1.5:

Evaluate $_6C_2$, and $_4C_4$.

Solution

Let's try $_6C_2$, or $\binom{6}{2}$:

$$_nC_r = \frac{6!}{(6-2)!2!}$$

$${}_nC_r = \frac{6!}{(4)!2!}$$

$${}_nC_r = \frac{(6)(5)(4)(3)(2)}{(4)(3)(2)(2)}$$

$${}_nC_r = \frac{(6)(5)}{(2)}$$

$${}_nC_r = \frac{30}{2}$$

$${}_nC_r = 15$$

Now let's tackle $\binom{4}{4}$:

$${}_nC_r = \frac{4!}{(4-4)!4!}$$

$${}_nC_r = \frac{4!}{0!4!}$$

The result of $0!$ is 1.

$${}_nC_r = \frac{4!}{4!}$$

$${}_nC_r = 1$$

This makes sense: there is only one way to choose four things from a group of four things. You choose all of them, and that is the only option.

Example 5.1.6:

How many 5-member committees are possible if we are choosing members from a group of 30 people?

Let's see: we have 30 people to choose from, so $n = 30$. We want to choose 5 members, so $r = 5$. Lastly, we don't care about the order in which we choose, so we use ${}_nC_r$:

$$\binom{30}{5} = \frac{30!}{(30-5)!5!}$$

$$\binom{30}{5} = \frac{30!}{25!5!}$$

$$\binom{30}{5} = \frac{(30)(29)(28)(27)(26)}{5!}$$

$$\binom{30}{5} = \frac{(30)(29)(28)(27)(26)}{120}$$

$$\binom{30}{5} = 142506$$

Example 5.1.7:

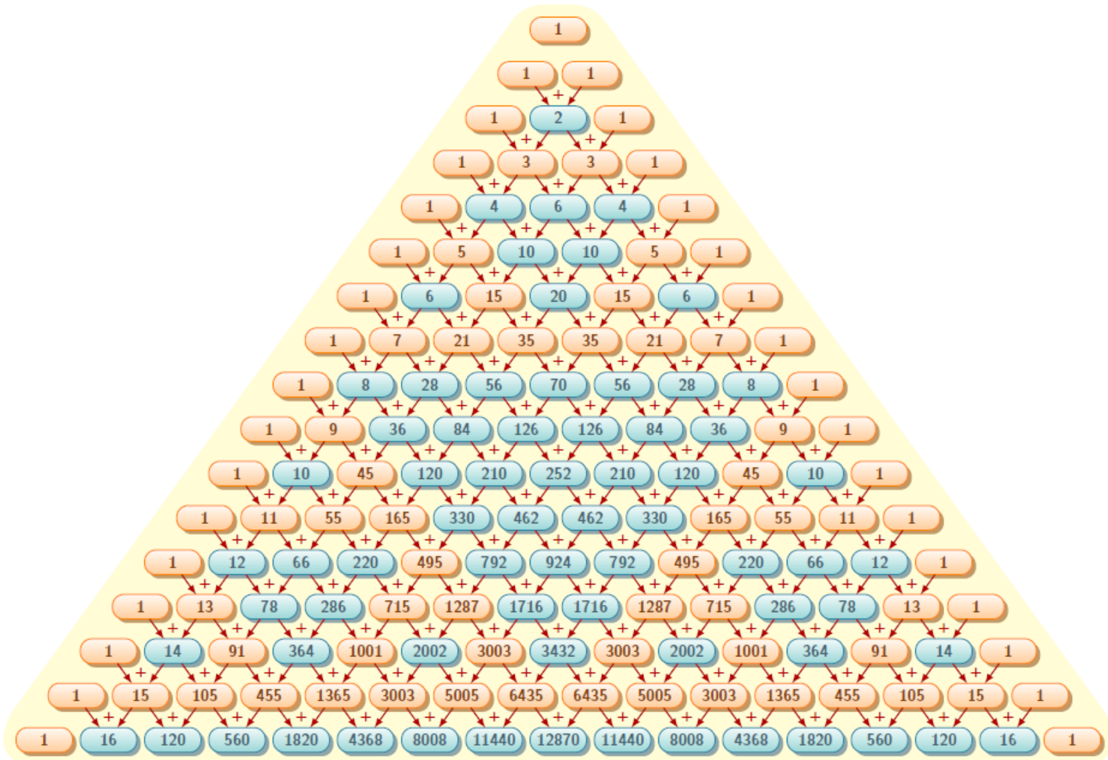
1. In how many ways can 3 men and 3 women sit in a row, if no two men and no two women are next to each other?
2. In how many ways can 3 men and 3 women sit in a circle, if no two men and no two women are next to each other?

Pascal's Triangle

Pascal's triangle was developed by the mathematician Blaise Pascal. It is generated by adding the two terms diagonally above to receive the new term, where the first term is 1, which is defined as ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$.

The triangle is useful when calculating ${}_nC_r$ as well: count down n rows, and then count in r terms. For example: ${}_7C_2$ means that we look at row 7, term 2: 6.

- Gives the coefficients of $(a+b)^n$.
- The entries of the nth row are $C(n,0), C(n,1) \ldots C(n,n)$
- The sums of each row are consecutive powers of 2.
- The third element from each row yields triangular numbers.



Binomial Expansion

$(x+y)^n = x^n + nx^{(n-1)}y + \cdots + \binom{n}{k}x^k y^{(n-k)} + \cdots + y^n$.

5.2: Probability: Living with odds

Probability is a subtle concept: There are several different things we mean by probable.

Randomness is also subtle. Ten heads in a row? Did we cheat? We probably did, but don't know yet as of when we typed this. In order to get ten heads in a row, we expect to have to cheat, but we don't know for sure that we will have to.

Our knowledge of things to come is also imperfect. What can we say in the face of imperfect knowledge? How can we reason knowing our knowledge is imperfect?

Definitions:

Outcome: An outcome (simple outcome) is the most basic possible result of observations or experiments.

Sample Space: Set of all possible outcomes.

Events: An event is one or more outcomes that share a property of interest

Example 5.2.1:

In a coin toss, the possible outcomes are T, H.

In tossing two coins the outcomes are TT, TH, HT, HH.

In rolling a standard die the outcomes are 1, 2, 3, 4, 5 or 6 dots.

In an experiment on colours of bred pea plants, the outcomes could be a red, white or pink plant.

Types of Probability:

Experimental: We observe over a length of time or perform an experiment many times and calculate the relative frequency of the event. The relative frequency is the number of times the desired outcome occurs per the number of times an experiment is performed or observations made – it's a percentage!

Theoretical: Based on a model where all outcomes are equally likely.

Subjective: Estimate based on intuition or experience (ideally to be made by an expert in the field who also has a sound grasp of probability).

Fundamentals of Probability

Example 5.2.2:

- When tossing two coins we might be interested in the event of two heads, or the event of at least one head.
- When rolling a standard die we might be interested in the event of rolling a one or the event of rolling an even number or the event of rolling a number less than 4.
- When breeding 6 pea plants we might be interested in the event all plants are the same colour.

Example 5.2.3:

Experiment: Toss two coins

Outcomes: TT, TH, HT, HH

Event: getting one head consists of TH or HT

Simple Calculations

Assumptions: a fair coin, a fair dice, well-shuffled cards

Method: Count the total number of possible outcomes n and count the number of outcomes in the event A.

$P(A) = \frac{|A|}{n}$.

In general, the probability of an event, A, occurring is $P(A) = \frac{|A|}{|S|}$, where $|S|$ is the total number of outcomes in the sample space S.

Formal Properties of Probability

Rules:

Let A, B, C, … be events. P(A) denotes the probability event A occurs.

RULE #1: $0 \leq P(A) \leq 1$ .for every event A. The probability of any event is a number between 0 and 1.

Rule # 2: P(E) = 0 if and only if the event is impossible.

Rule# 3: P(E) = 1 if and only if the event is a certainty.

Rule #4: $P(E^c) = 1 - P(E)$. (The probability of an event NOT occurring is 1 – probability the event occurs).

Thinking Out Loud:

Probability can also be expressed as a percentage. What is the range?

Example 5.2.4:

What is the probability of rolling a four on two six-sided dice?

There are 36 different ways of rolling two six-sided dice. (How do we know?)

There are 3 ways of rolling four (1+3, 2+2, 3+1)

So, the probability is 3/36 = 1/12

Thinking Out Loud:

What is the probability of flipping 10 coins and getting all the same?

What is the probability of flipping 10 coins and getting all heads?

Example 5.2.5:

From a well-shuffled deck of 52 cards, three cards are drawn at random. Find the probability that **exactly** one King will be drawn.

There are $\binom{52}{3}$ ways to select 3 cards out of 52. There are $\binom{4}{1}$ ways to select one King out of four Kings. Further, there are $\binom{48}{2}$ ways to select the remaining two non-King cards out of 48 non-King cards. Therefore, the probability that **exactly** one King will be drawn is

$$\frac{\binom{4}{1} \cdot \binom{48}{2}}{\binom{52}{3}} \tag{5.2.1}$$

.

Example 5.2.6:

From a well-shuffled deck of 52 cards, three cards are drawn at random. Find the probability that **at least** one King will be drawn.

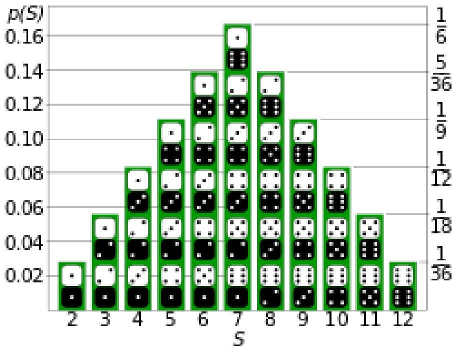
The probability is the sum of the probability that **exactly** one, two, and three King will be drawn. Hence ,

$$\frac{\binom{4}{1}\cdot\binom{48}{2}}{\binom{52}{3}}+\frac{\binom{4}{2}\cdot\binom{48}{1}}{\binom{52}{3}}+\frac{\binom{4}{3}\cdot\binom{48}{0}}{\binom{52}{3}} \tag{5.2.2}$$

.

Probability distribution

The probability distribution is a display of the probability for every possible event. For example, see the probability distribution of rolling two six-sided dice.



Combining Probabilities

Definition: Independent

Two events are independent if the outcome of one event does not affect the probability of the second occurring otherwise we say the two events are dependent

Example 5.2.7:

Tossing a coin twice, the event of the first toss being a head and the event of the second toss being a head are independent.

Tossing a coin twice, the event of the first toss being a head and the event of at least one head occurring are NOT independent.

Rules Continued:

RULE #5: If A and B are independent events then P(A and B) = P(A)P(B).

This rule generalizes to 3 or more independent events.:

Example 5.2.8:

Suppose you toss three coins. What is the probability of getting three tails? P(3 tails)=P(1tail) P(1tail) P(1tail)= 1/8.

Rules Continued:

RULE #6: If A and B are dependent events then P(A and B) = P(A)P(B given A)

{P(B given A) is the conditional probability of even B occurring when event A is assumed to have already occurred).}

Example 5.2.9:

Two members are selected from a pool of 17 male students and 23 female students. Find the probability that the first student selected is a male and the second is also male.

$P(A) =$ The first student selected is male.

$P(B) =$ The second student selected is male

$P(A) = \frac{17}{40}$

$P(B\,given\,A) = \frac{16}{39}$

$PA\,and\,B) = \left(\frac{17}{40}\right)\left(\frac{16}{39}\right)$

Definition: Mutually Exclusive

Two events are mutually exclusive (non-overlapping) if the two events can't occur at the same time. Two events are overlapping if they can occur at the same time.

Example 5.2.10:

Tossing a coin twice, the event of the first toss being a head and the event of the second toss being a head are overlapping. The outcome of HH would satisfy both events. Tossing a coin twice, the event of the first toss being a head and the event of no heads occurring are non-overlapping, since we can't have both occurring. When tossing a die, the event of tossing a number greater than 4 and the event of tossing an even number are overlapping events. An outcome of 6 would satisfy both.

Rules Continued:

Rule #7: If A and B are non-overlapping events then

P(A or B) = P(A)+ P(B)

This generalizes to 3 or more events.

Example 5.2.11:

Find the probability of rolling either a 1 or a 2 on a single die.

$P(1) = \frac{1}{6}$

$P(2) = \frac{1}{6}$

$P(1\,or\,2) = \frac{1}{6} + \frac{1}{6}$

$P(1\,or\,2) = \frac{2}{6} = \frac{1}{3}$

Rules Continued:

Rule #8: If A and B are overlapping events then

P(A or B) = P(A)+ P(B) - P(A and B).

Example 5.2.12:

Find the probability of drawing a queen or a heart from a standard deck.

$P(Q) = \frac{4}{52}$

$P(H) = \frac{13}{52}$

$P(Q\,and\,H) = \frac{1}{52}$

$P(Q\,or\,H) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$

$$P(Q\ or\ H)=\frac{16}{52}=\frac{4}{13}$$

Rules Continued:

Rule #9: Suppose the probability of an event in one trial is P(A). If all trials are independent, the probability that event occurs at least once in n trials is

$1-P\,(no\,event\ A\,in\,n\,trials)=1-P(not\ A\,in\,one\,trial)^n$

Example 5.2.13:

Find the probability of getting at least one 6 in five rolls of a single die.

More on conditional probability

We have seen that P(A and B) = P(A)P(B given A)

I.e. P(B given A) = P(A and B)/P(A).

P(B given A) = the (conditional) probability that event B occurs given that event A is known to have occurred.

Example 5.2.14:

if a coin was tossed twice, P(HH) =$\frac{1}{2}$.

P(HH given both coins showed same value) = $\frac{1}{4}$. (the condition tells us it was either HH or TT).

Rules Continued:

Rule #10: The odds for event A occurring are:

Odds for event $A=P(A):P(not\,A)$. This is usually converted to a pair of integers.

Example 5.2.15:

If you toss a coin twice, what are the odds you get at least one head appearing?

1:1 (50:50)

If the odds are 1:3 that a patient will survive heart surgery is that good or bad? What is it as a probability?

¼ = 0.25 = 25% not great

Example 5.2.16:

A fair coin is tossed 10 times. Which is more likely:

a. 10 heads in a row
b. THHTHTHTTH (giving 5 heads and 5 tails in total).

Both events are equally as likely- in each case, I explicitly predicted each toss of the coin! We confuse the fact that it is more likely to get 5 heads than 10 heads with a very specific outcome.

Coincidences

Partly because we don’t know probabilities as well as we should and partly because we over publicize uncommon events and attach significance to things after they happen we don’t appreciate coincidences properly.

When is your birthday: month and day?

It can be shown that in a room with 23 people there is a 50% chance two will share a birthday. In a room with 41 or more people, the probability is over 90%. (Note: if we specified in advance which day the birthday was on it would be far less likely to find two people with a birthday on that day).

Monty Hall Problem

Monty Hall runs a game show. Contestants pick one of three closed doors and win the prize behind the door. Behind one of the doors is a car, behind the other two doors, are pigs. After the contestant picks a door, Monty Hall opens one of the other doors to reveal one of the pigs. He asks the contestant if they would like to change their mind and take the other unopened door. Should they bother switching? Why?

5.3: Expected value

Definition:

For a probability distribution defined by $P(X=x)$, we define the expectation of the random variable X as

$$\begin{aligned} E(X) &= \sum_{i=1}^{i=n} x_i P(X=x_i) \\ &= x_1 P(X=x_2) + x_2 P(X=x_2) + \cdots + x_n P(X=x_n) \end{aligned}$$

where x_i represents the observed outcome and $P(X=x_i)$ is the probability of the outcome occurring.

The “expected value of X” can be interpreted as the mean value of X.

The expectation values can be considered in two ways.

1. Long-run average

This is the measure one would see if the experiment was repeated a large number of times, namely $E(X)=np$, where n is the number of times the experiment occurred and p is the probability for the event to occur.

Example 5.3.1

If we tossed a coin 1500 times, and the random variable X, represents the number of heads observed, we would expect 750 heads, that is $E(X)=750$.

2. Probability weighted average

This is the measure that takes into account the relative probabilities of each observed outcome.

Example 5.3.2

For the probability distribution with random variable X defined by

x	2	3	4	5
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = \sum_{i=1}^{i=4} x_i P(X=x_i) = (2\frac{1}{6}) + (3\frac{1}{6}) + (4\frac{1}{6}) + (5\frac{1}{6}) = \frac{15}{6}$.

Thus X has a mean value of $\frac{15}{6}$.

Example 5.3.3:

We toss 4 coins at the same time, then the probability of getting X number of tails:

x	0	1	2	3	4	Total
$P(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	

Then the expected value is

$E(X) = \sum_{i=0}^{i=4} x_i P(X=x_i) = (0\frac{1}{16}) + (1\frac{4}{16}) + (2\frac{6}{16}) + (3\frac{4}{16}) + (4\frac{1}{16}) = \frac{32}{16} = 2$.

Therefore, in the long run, we would expect to get 2 tails, when we toss 94 coins at the same time.

5.E: Basic Concepts of Probability (Exercises)

Exercise $5.E.1$: Counting

A typical PIN is a sequence of any seven symbols chosen from the 26 letters in the alphabets and the ten digits $0, 1, \ldots., 9$ with repetition allowed.

1. How many PIN's are available?
2. Supposing that symbols cannot be repeated, then how many PIN's are available?

Exercise $5.E.2$: Counting

How many (Canadian) postal codes would there be possible without repetition of letters or numbers?

Exercise $5.E.3$: Counting

Seven women and nine men are on the faculty in the Mathematics Department.

1. How many different committees are there which are made up of five members of the department, at least one of which is a woman?
2. How many ways can the five-person committee be arranged around a circular table?

Exercise $5.E.4$: Probability

You have 23 people in a room. What is the probability that the two of them have the same birthday?

Exercise $5.E.5$:Probability

A coin is tossed 7 times, find

1. the probability that we see at least 2 heads?
2. the probability that we see exactly 2 heads?
3. the probability that we see exactly 2 heads or exactly 4 heads?

Exercise $5.E.6$:Probability

What is the conditional probability that the sum of the dice is 10, 11, or 12 given that the first die rolled comes up a 6?

Exercise $5.E.7$:Probability

We have a standard deck of 52 cards and you select a card at random.

1. What is the probability you select a 5?
2. What is the probability you don’t select a 5?
3. What is the probability you select a 5 or 7? Are these mutually exclusive events or not?
4. What is the probability you select a 5 or a diamond? Are these mutually exclusive events or not?
5. What is the probability you select the Jack of Spades given you have selected a face card?
6. What is the probability you select a diamond given you have selected a red card?
7. What is the probability you selected a red card given you’ve selected a diamond?

Now suppose you select two cards from the deck.

8. What is the probability you select two Jacks assuming you replace a card and reshuffle the deck before selecting again? Are these dependent or independent events?
9. What is the probability you select two Jacks assuming you don’t replace a card once you’ve selected it? Are these dependent or independent events?
10. What is the probability you select a Jack and an Ace – again assuming replacement. Careful- you could select the Jack then the Ace or the Ace then the Jack.

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.8}$: Probability

In a town of 10,000 people, 400 have beards (all men), 4000 are adult men, and 5 of the townspeople are murderers. All 5 murderers are men and 4 of the murderers have beards.

Suppose you go to this town and select a towns person at random.

- Let A be the event that the person turns out to be one of the five murderers.
- Let B be the event the person is bearded.
- Let C be the event the person is an adult male.

Find P(A), P(A given B), P(B given A), P(A given not B), P(A given C), P(A given not C).

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.9}$: False Positive

Disease X is a disease affecting about 1 percent of the population. A test for Disease X will test positive on all afflicted with the disease and will also test positive for 5% of the population who do not have the disease.

1. What is the probability a randomly chosen person has disease X?
2. What is the probability a randomly chosen person will test positive for the disease?
3. Suppose you test positive for the disease. What is the probability you don't actually have the disease? (This is the conditional probability that you don't have the disease given you tested positive for it).
4. What would be the probability that you test positive for the disease twice given that you don't have disease X?
5. Can you identify (at least 2) criticism's of our theoretical probability calculations here?

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.10}$: Probability

1. the probability that we see at least 2 heads?
2. the probability that we see exactly 2 heads?
3. the probability that we see exactly 2 heads or exactly 4 heads?

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.11}$: Probability

A multiple-choice test has 10 questions, each with 4 possible answers. A student guesses all ten questions.

a. Find the probability that the student will get all ten questions right.
b. Find the probability that the student will get **at least 1** question right.

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.12}$: Probability

Suppose two fair dice are thrown.

1. What is the probability the sum of the dice is at most 4?
2. What is the probability the sum of the dice is more than 4?
3. What are the odds the sum of the dice is at most 4?
4. What is the probability the sum of the dice is at most 4 given that the first die shows a 3?

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.13}$: Combination

If $n \geq k+2$ and $k \geq 2$, show that $\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2}$ is even.

Exercise $\mathbf{5.}\boldsymbol{E}\mathbf{.14}$: Combination

Find the coefficient of x^5 in the binomial expansion of

$$\left(\frac{2}{x} + x^2\right)^{25} \tag{5.E.1}$$

.

Exercise $5.E.15$: Combination

For natural numbers n and r, $r < n$, show that

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{5.E.2}$$

Answer

under construction.

Exercise $5.E.16$:

A fair coin, a double-headed coin and a double-tailed coin are placed in a bag. A coin is randomly selected. The coin selected is then tossed.

1. Find the probability that the coin lands with a “head”.

2. When the coin is tossed, it lands “tail”. Find the probability that it is the double-tailed coin.

Answer

$\frac{1}{2}\frac{3}{4}$.

CHAPTER OVERVIEW

6: Introduction to Statistics

Learning Objectives

Develop the students:

- ability to study data
- statistical reasoning

Statistics is the science of collecting, analyzing, and drawing conclusions from data. There are two branches of statistics.

Branches of Statistics

Definition: Descriptive Statistics

Descriptive statistics is the branch of statistics used in describing the data via graphs, tables or other statistical measures. This is the type of statistics we do when we have a lot of data and want to summarize it appropriately.

Definition: Inferential Statistics

Inferential statistics is the branch of statistics that deals with inferring/estimating population characteristics from sample data. If a sample represents a given population accurately, then analyzing the sample can lead to significant conclusions about the population as a whole.

Terminology

Definition:Population

The population is the entire group that a statistical sample is drawn. If an accurate sample is drawn, significant hypotheses can be developed with reference to the entire population.

Definition: Sample

A sample is a set of data collected from a specific population by a defined procedure.

Common Sampling Techniques

- Simple random sampling
- Systematic sampling
- Convenience sampling, which is poorly named
- Stratified sampling

Note that one could use a combination of these methods – for example, stopping every 5^{th} person to participate in a survey is systematic and convenience sampling and all of these methods depend on randomness in some way or another!

Bias

A study suffers from **bias** if its design or conduct tends to favour certain results. It can happen as a result of failing to choose a truly representative sample **(selection bias/participation bias)**. Bias might also be present if the person conducting the study is biased (by having a personal stake in the study, by having strong beliefs or expectations on the subject), even if their bias is subconscious. It can even happen at the end of the study if the data is intentionally or unintentionally distorted to lead to a particular conclusion. Finally, there could be a flaw in the study conduct, resulting from a systematic measuring error for example.

Placebos

In an experiment, a **placebo** is a ‘phony treatment’ that is often given to the control group. On the surface (to a patient and possibly data collector) it appears identical to the treatment under study but it is missing the ‘active ingredient’ understudy. The **placebo effect** refers to the situation when patients improve just because they believe they are receiving a useful treatment.

Confounding Variables

Variables that are not intended to be part of a study that confound (confuse) a study’s results are called confounding variables.

Surveys and Opinion Polls

They are a type of observational study with their own special issues one should be aware of. Margins of error, confidence intervals, and confidence levels are often reported with opinion polls and survey results.

> **Definition: margin of error, confidence interval, and confidence level**
>
> The **margin of error** is a number or percentage which should be added and subtracted from the reported number in order to provide a range of numbers in which the actual number probably resides. This range of numbers is called the **confidence interval**. Margins of error and confidence intervals are always calculated with respect to a particular **confidence level**. Usually, the confidence level used is 95% (19/20). This means that we can be 95% confident that the confidence interval contains the correct value. The margin of error for 95% confidence is approximately equal to $\frac{1}{\sqrt{(N)}}$, where N is the size of the sample. As N increases, the margin of error decreases

Topic hierarchy

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6.1: Qualitative Data and Quantitative Data

Data

There are two types of data that we can collect:

- Qualitative data describes a subject, and cannot be expressed as a number.
- Quantitative data defines a subject and is expressed as a number (it can be quantified) that can be analyzed. There are two types of quantitative data continuous and discrete.

Example 6.1.1:

1. Ratings of a tv show
2. Grades of an exam.
3. Marks of an exam.
4. Students heights in a class

Graphs

There are many types, including:

1. Pie charts and bar graphs are used for qualitative data
2. Histograms (similar to bar graphs) are used for quantitative data
3. Line graphs are used for quantitative data
4. Scatter graphs are used for quantitative data

Graphs should contain:

- A descriptive title below the graph or chart
- A caption below the title (optional)
- Axes labelled with the name of variable, units (if applicable) and the variable intervals; intervals must be spaced according to scale
- A legend to indicate which data points belong to which set of data, if more than one data set is displayed

Example 6.1.2:

Summarizing Data

There are various ways to summarize a data set:

- Distribution tables
- Graphs of raw data
- Sample statistics such as mean, median, mode, standard error and standard deviation
- Graphs based on average values with error bars to indicate a standard error or standard deviation

Simple Descriptive Statistics

Descriptive statistics are numbers and processes that describe a group of data. The most common descriptive statistics focus on determining the "average" of the data. However, there is more than one "average," so we must be specific when finding them. Values which describe the “average” are:

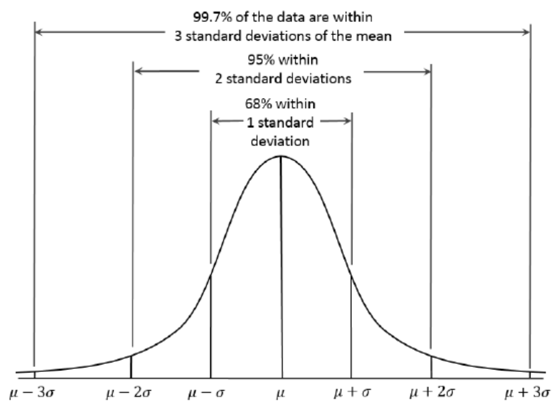
- Mean - the sum of all the data divided by the number of datum in the group, the "average" that most people mean (see what we did there?).
- Median - the middle-most datum, when all the data are arranged by the quantity
- Mode - the most common datum in the set

As you can imagine, data sets very rarely are all one value. Thus, we need to describe how the data is arranged around the mean. Values which describe the variation of the data around the mean are:

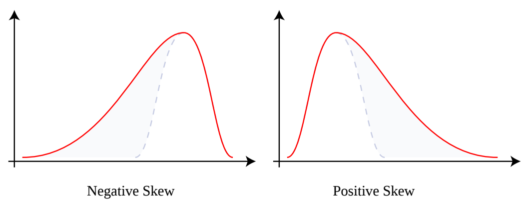
- Standard deviation describes, for a whole population, the dispersion of the whole population's data set.
- Standard error describes, for a representative sample's data set, the standard deviation of that set.

The Normal Distribution and Other Shapes of Data Distribution

Analysis of many phenomena results in a ***normal*** distribution of data. Normal distribution approximates a bell-shaped curve when data is plotted on a line graph. As the number of replicates in a data set increases, the graph approaches a perfect bell shape, so the mean, median and mode are all at the peak of the curve



A distribution may be skewed due to a disproportionate number of extremely high or low values, especially if the sample size is small.



A distribution may show more or less variation, around the mean. This is quantified by the size of the standard deviation of the distribution.

It is important to remember that not all types of data distributions have a single peak.

6.2: Descriptive Statitics:Measures of Center, Measures of Variation and the Five - Number Summary

Measures of Central Tendency

Definition: Mean, Median, Mode

- **Mean**: Add each data value and divide by the number of data values.
- **Median**: Arrange the data values in numerical order. The median is the middle data value. If there is an even number of data, then find the mean of the two closest to the middle.
- **Mode**: The data value that occurs most often.

Example 6.2.1:

Given the following set A, determine the mean, median, and mode of the set.

$A = 1, 1, 1, 2, 3, 5, 5, 7, 9, 12, 23$

$Mean = \frac{\Sigma A}{n}$, or the sum of all terms of A divided by the number of terms in A

$= \frac{1+1+1+2+3+5+5+7+9+12+23}{11}$

$= \frac{69}{11}$

$= 6\frac{3}{11}$

$Median = 5$

$Mode = 1$

Measurements of Variation

Measurements of variation are well named. These quantities describe how far apart the data points can be from each other. If a data set is imagined as a bull's eye, measurements of variation will describe the size of the target, as well as where there are groups of points or gaps in points. We use the following measures to describe the dispersion of data:

Definition: Measures of Dispersion

- **Range** describes the span of the data, or how far apart the biggest and smallest values are. It is calculated by subtracting the minimum value from the maximum value
- **Clusters** occur when groups of data occur together, and apart from the rest of the data points. There may be one or more clusters in any given data set.
- **Gaps** are places where data is expected to occur but does not.
- **Outliers** are data points which occur individually and do not behave according to the trend described by the rest of the data.
- **Standard Deviation**, given by s, describes how far, on average, observed data is from the expected mean.

Where n is the number of observations, we can determine a number of quantities:

Definition: Describing the Data

Variance describes how far from the average a set of values in a data set is expected to fall. Variance = s^2

The **first quartile** (**Q_1**) is the median of the part of the entire data set that lies at or below the median of the data set.

The **second quartile** (**Q_2**) is the median of the data set.

The **third quartile** (**Q_3**) is the median of the part of the entire data set that lies at or above the median of the data set.

Interquartile Range describes the difference between the first and third quartiles. IQR = $Q_3 - Q_1$

Five-Number Summary consists of five numbers that describe a data set:

1. The data's minimum value
2. The first quartile
3. The median
4. The third quartile
5. The data's maximum value

There are many ways in which a set of data can be distributed. In this course, we will focus on five distributions: uniform, skewed to the right, skewed to the left, bimodal, and normal.

Example 6.2.2:

Given data set B, give the five-number summary of the set.

$B = 1, 1, 2, 2, 4, 5, 5, 6, 6, 6, 6, 7, 8, 8, 9, 9, 9, 10, 12, 14, 16, 22, 29$

Remember, the five-number summary of a set is

1. The data's minimum value
2. The first quartile
3. The median
4. The third quartile
5. The data's maximum value

Let's start with the minimum, maximum, and median values, as those are the simplest.

1. The minimum value is 1.

3. The median value is the twelfth value (there are 23 values in all): 7

5. The maximum value is 29.

For the first and third quartiles, things get a little more complicated. When determining the first quartile, we include the median. When we switch to the third quartile, however, we cannot use the median again. Whether we use the median for the first or third quartiles is an arbitrary choice, but since Microsoft Excel uses the median to determine the first quartile, that's what we will do. Not all software does this, so you should be aware that things might not always be the same.

2. The first quartile is the average of the sixth and seventh terms (there are 12 terms in the first half, median included): $\frac{5+5}{2} = 5$

4. The third quartile is the 18th term (there are 11 terms in the second half, median excluded, and 12 plus six is 18): 10

So, the five-number summary is:

$1,\ 5,\ 7,\ 10,\ 29$

Exercise 6.2.1:Five number summary

Given $3, 2, 0, 5, 5, 3, 1, 0, 3, 2$

Obtain the five-number summary for these data.

a. Identify potential outliers, if any.
b. Construct a boxplot.

Note:

Wh**at is the meaning of the interpolated median? How does it differ from the median?**

6.3: Introduction to Statistical Calculations using Microsoft EXCEL

Example 6.3.1: In Class project.

We will collect the data in class.

Instructions:

1. Open a new Excel worksheet and save as Lab1.xls – I'd like the finished version submitted to the Blackboard site digital drop box. Put your name (first and last) in Cell A1. If you choose to work with a partner put the second name in Cell A2 and only submit one worksheet between you.

Use the appropriate decimals button to reduce all answers to at most 3 decimals. Save often!

Use Excel functions to calculate:

a. The mean
b. The median
c. The mode, Explain your mode answer!
d. The standard deviation
e. The range
f. The interquartile numbers
g. The Five number summary.

The functions you'll need are the average(cell range), median (cell range), mode (cell range), max(cell range), min(cell range), stdev(cell range), Quartile (cell range, quartile number). Try using the function button to access the functions instead of typing them out.

There is an easier way to get the above information and more!

2. Go to the file Button (top left hand corner), Excel Options, Add-Ins. Select the Data Analysis Tool Pack and add it in!

- Highlight the data from Question 1.
- Under Data select data analysis, descriptive statistics. Enter the data cell range and for input just put a cell where you want the top line to appear. Check the box for summary statistics and hit enter!

3. To draw box plot: calculate the
statistical functions QUARTILE(q1), MIN, MEDIAN, MAX and QUARTILE(q3)
in that order for each data set. Arrange the results on an Excel
worksheet.

6.4: Binomial distribution and Normal Distribution

Discrete probability distributions

Binomial Trials

1. There are a fixed number of independent trials n.
2. Each trial has only two (hence binomial) outcomes, either "success" or "failure".
3. For the trials, the probability of success,p is always the same, and the probability of failure, $q = 1 - p$, is also always the same.
4. The expected value $E(X) = np$.

Excel Activity

Goal: Get a "feel" for binomial distributions by finding their probability distribution tables and graphing them.

Calculate the probability distribution table for X, a binomial distribution with 10 trials and probability of success p = 0.02. Use the drag feature to save yourself from a lot of typing!

X	P(X = x)
0 (say this is in cell A2)	=BINOMDIST (A2, 10,0.2,False)
1	
2	

Use Chart Wizard to plot the probabilities as a histogram (bar chart with no gaps!) You'll need to click on the bars of the chart and Select Data to get the 0, 1, 2, … as the X-axis labels and you'll need to select Format Data Series to remove gaps.

Repeat for n=10, p= 0.5 and n=10, p = 0.9. You'll get 3 tables and 3 histograms. What are the shapes of each distribution?

Answer the following: For small n Binomial Histograms tend to be ______ skewed if p < 0.5 and ______ skewed if p > 0.5.

Continuous probability distribution

6.E: Introduction to Statistics (Exercises)

CHAPTER OVERVIEW

7: Rational Reasoning

Learning Objectives

Develop the students:

- rational reasoning
- ability to work with Egyptian fractions.

Topic hierarchy

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7.1: Dimensional Analysis

Thinking out loud

Small pizza is 8" and large Pizza is 16". Are we getting the same amount of pizza if we order two small pizzas instead of one large one?

We all have faced a situation when need to be able to change from one unit of measurement to another unit of measurement. In this section, we discuss a method of converting units called dimensional analysis.

1 day = 24 hours,

1 hour = 60 minutes, and
1 minute = 60 seconds.

Imperial measurement

1 inch

1 foot= 12 inches

1 yard= 3 feet=36 inches

1 mile =5289 feet

Metric

millimeter (mm)

centimeter (cm), 1 cm = 10 mm

meter (m) , 1m= 100 cm= 1000 mm

kilometer (km). 1 km= 1000 m

Definition

A unit ratio is a fraction that has a value of 1 if both the numerator and the denominator are expressed in the same units.

Example 7.1.1:

If the speed limit says 90 kilometres per hour, what is the speed in miles per hour?

We can use dimensional analysis to convert this speed to miles per hour. We can also use reasoning to deduce that we need to divide 90 by 1.6.

Area

Example 7.1.1:

John and James have decided to pull up their old carpet and buy a new carpet. The room measures 15 feet by 11 feet, so the area is 165 square feet. However, when they go to the carpet store, they find that the prices are in square yards. How many square yards is their floor?

Maximizing area

Temperature

Temperature: In 1714, a German instrument maker named Gabriel Fahrenheit made the first mercury thermometer. He designated the lowest temperature he could create in the laboratory as 0° and the normal temperature of the body as 98°. On his scale, the freezing point of water is 32° and the boiling point of water is 212°.

Metric system

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7.2: Egyptian Fractions

This page is a draft and is under active development.

The Egyptians used a pictorial number system, with different symbols for every power of 10 to 1 000 000.

alt alt

1 = |

4 = ||||

10 = ∩

23 = ∩ ∩|||

100 =

112 = ∩||

Egyptian fractions can only have 1 as the numerator: $\frac{1}{4}, \frac{1}{16}, \frac{1}{137}$ etc. This can also be called unit fractions. (They also used $\frac{2}{3}$ but we will ignore this for now). The unit fraction is made by writing the number with a "mouth" symbol over the top –

$\frac{1}{2} =$

Egyptians did not like repeating fractions, therefore, each fraction must be unique. As a result, any fraction with numerator > 1 must be written as a combination of some set of Egyptian fractions. As a result of this mathematical quirk, Egyptian fractions are a great way to test student understanding of adding and combining fractions with different denominators (grade 5-6), and for understanding the relationship between fractions with different denominators (grade 5). We can use manipulative to explore Egyptian fractions. For a detailed activity, please see Activity.

Eye of Honus

alt

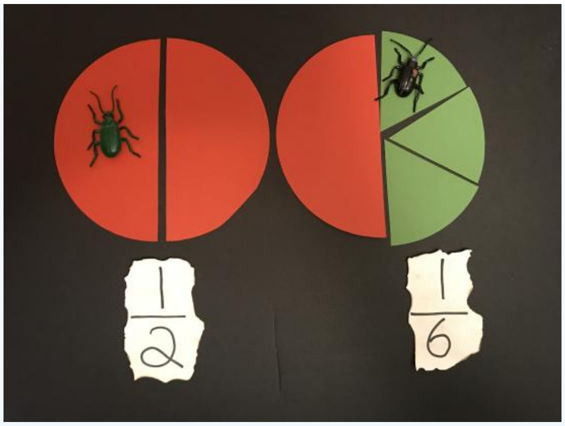
Party Time

Example 7.2.1: Party Time

You're having a pie party. Unfortunately, you forgot to cut the pies before the party started. How do you cut and split the pies evenly amongst your guests, if there are 3 people and 2 pies $\frac{2}{3}$? How do you split the pies using Egyptian fractions?

Solution

Using our manipulative, we begin by laying out 2 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ of a pie for each person and $\frac{1}{2}$ of a pie leftover. We can split this last half piece into thirds again, giving us $\frac{1}{6}$ of a pie for each person. That is $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.



Example 7.2.2:

If we were asked to write $\frac{3}{5}$ into a sum of Egyptian fractions, we could write $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$, since each fraction has a numerator of 1. The only problem with this is that they aren't unique fractions, therefore it wouldn't work for the sum of Egyptian fractions.

$\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$.

Example 7.2.3:

What if there were only 4 pizzas to be split amongst 8 friends?

$\frac{5}{8}=\frac{1}{2}+\frac{1}{8}$.

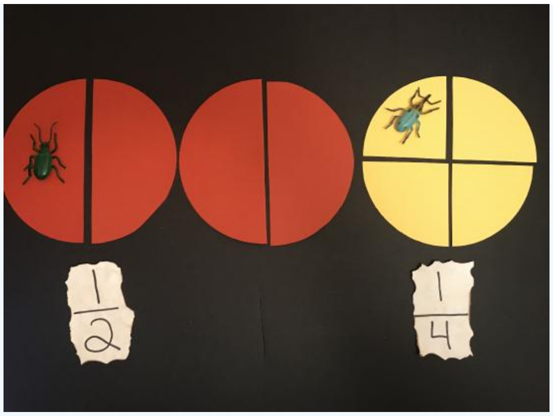
Exercise 7.2.1:

You're having a pie party. Unfortunately, you forgot to cut the pies before the party started. How do you cut and split the pies evenly amongst your guests?

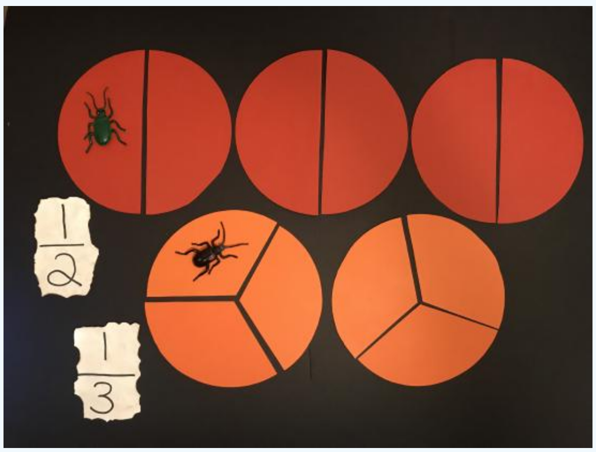
1. There are 4 people and 3 pies $\frac{3}{4}$. How do you split the pies using Egyptian fractions?
2. There are 6 people and 5 pies $\frac{5}{6}$. How do you split the pies using Egyptian fractions?

Answer

1. Using our manipulative, we begin by laying out 3 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ a pie for each person and a whole pie leftover. We can split this pie into 4 equal pieces, giving us $\frac{1}{4}$ of a pie for each person.



2. Using our manipulative, we begin by laying out 5 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ a pie for each person and two whole pie leftovers. We can split these two pies into 6 equal pieces, giving us $\frac{1}{3}$ of a pie for each person.



Bigger Parties, Bigger Problems

You are now having an even bigger party with a lot more pies. Can you find the sum of Egyptian fractions that will help split the pies evenly amongst your guests? Having manipulative for these next questions would be unrealistic as there would be too many pies. Instead, we can think about the factors of 100 and use these as our denominators of our Egyptian fractions.The factors of 100 are $1, 2, 4, 5, 10, 20, 25, 50,$ and 100.

Example 7.2.4:

There are 100 people and 91 pies $\frac{91}{100}$. How do you split the pies using Egyptian fractions?

Solution

Looking at our factors of 100, we can begin by subtracting the biggest Egyptian fraction $\frac{1}{2}$.

$$\frac{91}{100} - \frac{1}{2} = \frac{91}{100} - \frac{50}{100} = \frac{41}{100} \quad .$$

Next, we subtract $\frac{1}{4}$.

$$\frac{41}{100} - \frac{1}{4} = \frac{41}{100} - \frac{25}{100} = \frac{16}{100} \quad .$$

Notice that we can't subtract Next, we subtract $\frac{1}{5}=\frac{20}{100}$. Therefore, we subtract $\frac{1}{10}$.

$\frac{16}{100}-\frac{1}{10}=\frac{16}{100}-\frac{10}{100}=\frac{6}{100}$.

Next we subtract $\frac{1}{20}$.

$\frac{6}{100}-\frac{1}{20}=\frac{6}{100}-\frac{5}{100}=\frac{1}{100}$.

Thusm $\frac{91}{100}=\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{100}$. Each person would get $\frac{1}{2},\frac{1}{4},\frac{1}{5}$. and $\frac{1}{100}$ of a pie.

Exercise 7.2.2:

There are 100 people and 99 pies $\frac{99}{100}$. How do you split the pies using Egyptian fractions?

Answer

$\frac{99}{100}=\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{25}$. Also note that it can be spilit into Egyptian fractions which do not have their denominators as factors of 100. $\frac{99}{100}=\frac{1}{2}+\frac{1}{3}+\frac{1}{10}+\frac{1}{20}+\frac{1}{150}$.

Theorem 7.2.1

Any fraction $\frac{2}{n}$, where n is odd, can be expressed as a sum of two Egyptian fractions.

Proof

coming soon.

- Thanks to Jillian Periliat for all the diagrams.

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7.E: Rational Reasoning (Exercises)

Exercise $7.E.1$

Convert 6 feet 4 inches to centimetres. Fact: 1 inch $=2.54$ cm

Answer

$76 * 2.54 \text{cm}$

Exercise $7.E.2$

Convert 850 square feet to square meters. Why do you think that Real-estate sites like to list house area regarding square feet instead of square meters? Fact: 1 foot = 0.3048m

Exercise $7.E.3$

John claims that the surface area of a cone is given by the formula: $A = \pi r \sqrt{r+h}$

where r is the radius of the cone and h is the height of the cone. How can you convince John, that he must be wrong without resorting to showing her formula in a textbook?

Exercise $7.E.4$

Yahoo Autos cites that the fuel efficiency of the 2008 Toyota Prius is 4L/100 km in the City and 4.2 L/100 km on the highway. American site states that Toyota's 2008 Prius hybrid car uses an average of 48 miles per gallon in city driving, and 45 mpg on the highway. Do these figures agree? According to the Canadian figures, if I spend about $20 per week in a city driving in a 2008 Prius, roughly how many kilometres have I travelled?

Exercise $7.E.5$

At what temperature do Celsius and Fahrenheit agree?

Exercise $7.E.6$

The exterior dimensions of a freezer are 48 inches by 36 inches by 24 inches, and it is advertised as being 27.0 cubic ft. Is the advertised volume correct?

Exercise $7.E.7$

Which holds more soup: a can with a diameter of 3 inches and a height of 4 inches or a can with a diameter of 4 inches and a height of 3 inches?

Exercise $7.E.8$

A larger cube has a volume of 81 m^3. A smaller cube has the length of the edges one-third of the length of the edges of the larger cube. What is the volume of the smaller cube?

Exercise $7.E.9$

A larger equilateral triangle was created using four smaller equilateral triangles as shown in the figure. The perimeter of the smaller triangle is 18 cm, then what is the perimeter of the larger equilateral triangle?

Exercise $7.E.10$

There are 100 people and 97 pies $\frac{97}{100}$. How do you split the pies using Egyptian fractions?

Index

Glossary

Sample Word 1 | Sample Definition 1

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    - 4.2: 2-D Geometry - *CC BY-NC-SA 4.0*
    - 4.3: 3-D Geometry - *CC BY-NC-SA 4.0*
    - 4.4: Transformations - *CC BY-NC-SA 4.0*
    - 4.5: Symmetry - *CC BY-NC-SA 4.0*
    - 4.6: Summary - *CC BY-NC-SA 4.0*
    - 4.E: Basic Concepts of Euclidean Geometry (Exercises) - *CC BY-NC-SA 4.0*
  - 5: Basic Concepts of Probability - *CC BY-NC-SA 4.0*
    - 5.1: Counting - *CC BY-NC-SA 4.0*
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  - 6: Introduction to Statistics - *CC BY-NC-SA 4.0*
    - 6.1: Qualitative Data and Quantitative Data - *CC BY-NC-SA 4.0*
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    - 6.3: Introduction to Statistical Calculations using Microsoft EXCEL - *CC BY-NC-SA 4.0*
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    - 6.E: Introduction to Statistics (Exercises) - *CC BY-NC-SA 4.0*
  - 7: Rational Reasoning - *CC BY-NC-SA 4.0*
    - 7.1: Dimensional Analysis - *CC BY-NC-SA 4.0*
    - 7.2: Egyptian Fractions - *CC BY-NC-SA 4.0*
    - 7.E: Rational Reasoning (Exercises) - *CC BY-NC-SA 4.0*
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Suggested further readings

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