Does Identity Have a Sense?

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January 30, 2024

Plan :

G. Frege *Über Sinn und Bedeutung*(1892)

Definitions / Linguistic Conventions :

- (1) Phosphorus (gr-lat) = Lucifer (lat.) = Morning Star ;
- (2) Hesperus (gr-lat) = Vesper (lat.) = Evening Star

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Astronomical fact : (3) Phosphorus = Hesperus

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Frege *knows* that Phosphorus is a body illuminated by the Sun *if and only if* Frege *knows* that Hesperus is a body illuminated by the Sun.

WRONG ! (unless Frege knows that Phosphorus = Hesperus) : the problem of *intensional contexts* in the presence of propositional attitudes

Frege's solution : Sense and Reference

- (1) Phosphorus (gr-lat) $:=$ Lucifer (lat.) $:=$ Morning Star ;
- (2) Hesperus (gr-lat) $:=$ Vesper (lat.) $:=$ Evening Star ;
- (3) Phosphorus $=$ Hesperus ;
- (4) triangle $=$ trilateral but triangle $:\neq$ trilateral.

$$\frac{a := b}{a = b}$$

but NOT

$$\frac{a = b}{a := b}$$

G. Frege *Grundgesetze der Arithmetik* (1893-1903)

Die Identität ist eine so bestimmt gegebene Beziehung, dass nicht abzusehen ist, wie bei ihr verschiedene Arten vorkommen können

Identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur.

Cf. P. Geach, theory of *relative identity*.

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- endurance versus perdurance ;
- Theseus Ship and its likes.

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- equivalence relation (reflexive, symmetric, and transitive) **InId** ;
- that satisfies **InId** ;
- whether or not the converse principle of *Identity of Indiscernibles* (**IdIn**) holds is a matter of continuing debate...

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- These formal tools support a philosophical analysis of identity concept ;
- Question : Do these formal tools also restrict such an analysis leaving aside alternative ways to construe the identity concept ?
- Yes, I think so.

Methodological Remark

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- analysing linguistic examples ;
- using standard formal tools : First- and Second-order Classical logical calculi and their modal extensions.

This the core method of Analytic Philosophy after the Linguistic Turn. In my opinion this approach is too limited.

Methodological Remark (continued)

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Today's physics provides interesting new insights on the identity concept (cf. the case of bosons and fermions). In this talk I leave physics and all science aside and only draw on some novel approaches in the pure mathematics.

Structural Identity in the Bourbaki-style mathematics

Mathematical practice of the 20th century provides reasons to identity certain *isomorphic* structures. This is a major motivating idea of Mathematical (and also non-mathematical) Structuralism.

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Def. : Groups $< G, \circ >, < G', \odot >$ are called isomorphic if there is a bijective map f between their underlying sets

$$f : G \xrightarrow{\sim} G'$$

such that for all g_1, g_2 from G we have $f(g_1 \circ g_2) = f(g_1) \odot f(g_2)$

Desideratum : Equivalence principle for isomorphisms

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Problem (P. Benacerraf) : **EPI** is <u>not</u> formally supported by the standard set-theoretic foundations of mathematics where the base set theory (typically ZF) includes the standard identity relation $=$ construed as indicated above. In this setting it is straightforward to provide examples of isomorphic structures with certain different properties.

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Benecerraf's example : finite ordinals (natural numbers) construed either in Zermelo's way as $\{\ldots\{\emptyset\}\ldots\}$ or in von Neumann's way as $\{\ldots\{\emptyset,\{\emptyset\}\},\ldots\}$.

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Two forms of identity : judgemental (aka definitional) and propositional :

- Judgement $P \equiv_T Q$ asserts that terms P, Q' of the same type A are equal. (Terms of different types cannot be compared)
- Proposition $P =_T Q$ is a type ; Judgement $p : P =_T Q$ asserts that proposition $P =_T Q$ has proof p and thus is (intuitionistically) true

Two forms of identity in MLTT (continued)

MLTT validates the following rule, according to which a judgemental equality entails the corresponding propositional equality :

$$\frac{P \equiv_T Q}{refl_P : P =_T Q} \tag{1}$$

where $refl_P$ is the canonical proof of proposition $P =_T Q$ called the *reflection* of term P. But not the converse rule.

Tower of identity types

$$p, q : P =_T Q$$

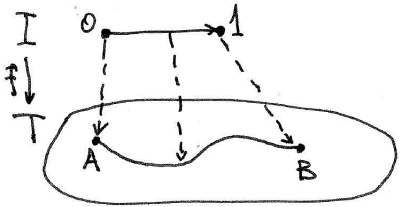
$$p', q' : p =_{P =_T Q} q$$

$$\ldots$$

Homotopy theory : paths

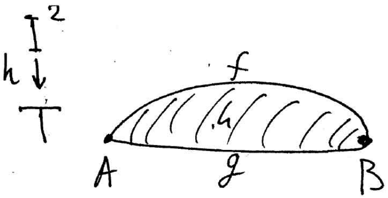
Def. : A path is a continuous map $p : I \to S$ from some distinguished *unit space* I (usually $[0, 1] \in \mathbb{R}$) into base space S.

If the unit space is thought of as a (directed) time unit then path p represents a continuous motion of a test point that begins at point $A = p(0)$ and ends at point $B = p(1)$. Beware that the same curve with endpoints A, B may represent different paths p, q.



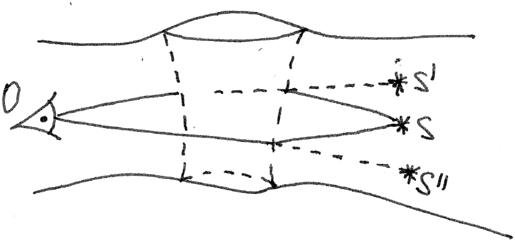
Homotopy theory : path homotopy

Idea : A homotopy is a path between paths
Def. : A homotopy is a continuous map of form $h : I^2 \to S$.



Non-homotopic paths : example of gravitational lensing with a wormhole

Whether two given paths between fixed points are homotopic or not depends on the topology of the base space.



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Remark : Paths, homotopies and higher homotopies are *directed* (since the unit interval is directed) but always *invertible* : given a path p from A to B there always exists the inverse path p^{-1} from B to A : for every continued motion there exists a unique backward motion.

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- paths are (partially) composable : given path $A \rightsquigarrow B$ and path $B \rightsquigarrow C$ there exist composed path $A \rightsquigarrow C$, which is uniquely defined up to homotopy (mind the problem of gluing the two time intervals into one of the same unit duration !) ;

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- points, paths and their composition form an algebraic structure of *groupoid* (since the paths are invertible !) called the fundamental groupoid of the underlying space ; by collapsing all points and identifying homotopic paths, one gets a group of loops, which is the *fundamental group* of the underlying space (Poincaré 1895) ;

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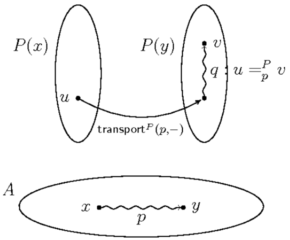
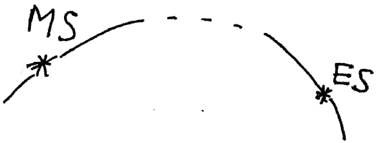
InId in HoTT : Transport of structures

(from Ledent&Wiedijk 2014)



Application to Frege's example

The Morning Star is the Evening Star. In order to establish this it is sufficient to observe or theoretically reconstruct a continuous trajectory from one to the other.



Identity through Time and Change

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The transport of properties and of structures along paths preserves them (up to homotopy of the required level) but still allows for distinguishing the base terms (= "temporal stages") by other means, formally, via their definitions. Recall that the propositional identity does not entail the definitional (judgemental) identity.

Endurance versus Perdurance

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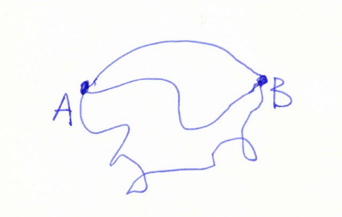
Cf. the definitions of Morning Star and Evening Star : on the pain of contradiction a star cannot appear both in the morning and in the evening simultaneously (since the morning is not the evening and vice versa : Morning $\neq$ Evening !) — but it can at different times.

Higher identity types in HoTT

While at the propositional level the (propositional) identity of given terms is a relation — it either holds or does not hold — the higher identity types have a more complex structure that may involve multiple paths, multiple homotopies between the paths, etc. all the way upward.



Quantum paths ?

Relation versus Structure (Voevodsky)

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while relations (as propositions) either holds or does not hold, structures, generally, are more complex. A structure can be reduced to relation by ignoring all its details save the fact that it is not empty (propositional truncation).

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MLTT + UA imply EPI

Some Traditional Philosophical Questions about Identity in the Standard Formal Setting

Some Reasons to Learn more from Science and Mathematics

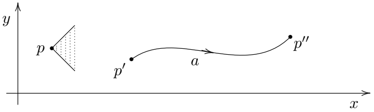
Identity in HoTT

Identity in DTT

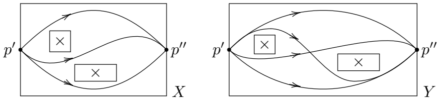
Conclusion :

Directed Spaces, Directed Homotopy theory (M. Grandis 2001)

Idea : A space (or rather spacetime) where paths, generally, are not invertible : a model of non-reversible worlds



$$(x, y) \leqslant (x', y') \quad \Leftrightarrow \quad |y' - y| \leqslant x' - x.$$



(from Grandis 2000)

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- identity in DTT is a spatiotemporal structure (not a relation) that is, generally, non-reversible ;
- The spatio-temporal structure is that of objects of a given type. Cf. *No entity without identity* motto (Quine).

Theseus Ship Resolved

$$\begin{tikzcd} & S_1 \\ S_0 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & S_2 \end{tikzcd}$$

$S_1 = S_0$ and $S_2 = S_0$ but $S_1 \neq S_2$

(also $S_0 \neq S_1$ and $S_0 \neq S_2$)

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Tentative answer : yes. Isn't too flexible and too soft ? The hard "ultimate" universal all-purpose identity concept may be well a metaphysical illusion.

Thanks !