Negation in Kolmogorov's Calculus of Problems

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Plan:

1. Problems and Theorems according to Kolmogorov
2. Negated and Unsolvable Problems
3. QHC
4. Conclusions

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Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Es gilt dann die folgende merkwürdige Tatsache: Nach der F...

Calculus of Problems 1932:

(1) "Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [. . .] If we can reduce the solution of problem b to the solution of problem a, and the solution of problem c to the solution of problem c to the solution of problem b, then the solution of c can also be reduced to the solution of a.

Calculus of Problems 1932:

(2) "The following remarkable fact holds: the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting" [reference to *Die formalen Regeln der intuitionistischen Logik*, 1930, in two parts]

Calculus of Problems 1932:

(3) "[Provided that] the general intuitionistic presuppositions are accepted . . . the intuitionistic logic . . . should be replaced by the calculus of problems, since the objects under consideration are in fact problems, rather than theoretical propositions. [the emphasis is mine]

Kolmogorov-Heyting controversy

Question: Is the difference between Kolmogorov's proposed interpretation of Heyting's calculus (= intuistionistic propositional calculus) in terms of *problems* and Heyting's own interpretation of this calculus in terms of *propositions* (Deutsch : *Aussagen*) essential or it is merely linguistic and superficial?

Kolmogorov-Heyting controversy

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Heyting 1958: "The older interpretations by Kolmogoroff (as a calculus of problems) and Heyting (as a calculus of intended construction) were substantially equivalent."

The intuitionstic notion of proposition

I leave out here some historical details and explain here this intuitionistic notion following Per Martin-Löf (1984):

"A proposition is defined by laying down what counts as a proof of the proposition. ... A proposition is true if it has a proof, that is, if a proof of it can be given."

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“When we hold a proposition to be true we make a *judgement*.”

Kolmogorov to Heyting, October 12, 1931

Every proposition p in your conception is, in my view, one of these two sorts:

α) p expresses the expectation that in such and such circumstances a [mathematical] experiment will give a determined result (for exemple, that an attempt to decompose an even number n into a sum of two prime numbers p, q will be successful if all pairs (p, q), where $p < n$ and $q < n$, are used. Every such "experiment" should be, of course, realisable with a finite number of well-defined operations.

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β) *p* expresses an intention of finding a construction. ...

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I prefer to reserve the name of proposition (Aussage) only to propositions of type α) and call "propositions" of type β) simply problems. Given a proposition p one has problems $\neg p$ (to reduce p to contradiction) and $+p$ (to prove p).

Kolmogorov's comment of 1985:

"On the interpretation of intuitionistic logic" was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types - propositions and problems.

Negated Problems according to Kolmogorov 1932

"$\neg A$ is this problem: assuming that a solution of A is given, to derive a contradiction." (Similar to Brouwer-Heyting negation.)

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Question: Contradiction is, *prima facie*, a propositional notion. If contradiction is understood as usual then the above definition is in odds with Kolmogorov's strategy to keeping propositions and problems apart. To avoid this difficulty Sanz proposes a counterpart of the standard *falsum* of propositional logic ($\bot$), that he calls a *panacea* : a problem that implies a solution of any given problem (by analogy with *ex falso*).

Negated Problems, continued

A footnote follows:

"We note that $\neg A$ should not be read as "prove the unsolvability of problem A". In the general case, if the "unsolvability of problem A" is considered as a completely defined notion, we only obtain that $\neg A$ implies the unsolvability of A but not the converse assertion. If, for example, it were proved that a realization of the well-ordering of the continuum is beyond our possibilities, it would not be possible to assert that the existence of such a well-ordering implies a contradiction. "

Negated Problems: interpretations

- Coquand (2007): the existence of well-ordering of the continuum (or the existence of some other mathematical construction) can be not provable in some theory (say, ZF) but still be consistent with this theory
- Melikhov (2017): Kolmogorov "conflates implications [i.e, reductions] between problems with implications between propositions", which makes the footnote ambiguous.

Negated Problems: my interpretation

An easier example: the trisection of a given angle with a ruler and compass (problem T.) Now by *solution* of T one may understand either

1. the two wanted lines trisecting the given angle, or
2. a general method of constructing such lines with ruler and compass for any given angle.

Negated Problems: my interpretation, contd.

Clearly only (2) but not (1) corresponds to the usual notion of what is an expected solution of T. But observe that (1) is on par with the intended solution of problem from the above Kolmogorov's example (the well-ordering of the continuum), where no specific constructive means are specified.

As we know it today via the Galois theory, T is *unsolvable* in the sense that, provably, no solution of T of type (2) exists. In other words, (a hypothesis of) the existence of (2) leads to contradiction.

Negated Problems: my interpretation, contd.

But the existence of solution in the sense (1) is intuitively obvious and does not lead to contradiction. Now, if we understand "T is unsolvable" as usual (i.e. as above) but define $\neg T$ as "the existence of solution (1) leads to contradiction" we get the situation described in the above Kolmogorov's remark: $\neg T$ does not hold but T is nevertheless unsolvable.

This example strongly suggest to redefine $\neg T$ as "the existence of solution (2) leads to contradiction". In this case $\neg T$ and "T is unsolvable" become equivalent.

Kolmogorov's critique of the B-H intuitionistic negation

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Cf. the Negationless Intuitionistic Logic by G.F.C. Griss, see Th. Ferguson 2023

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"Therefore the major result of the intuitionistic criticism of negated propositions should be formulated in the following simple way: in general is is meaningless to consider the negation of a general proposition as a definite proposition. But then the subject of intuitionistic logic disappears, since the law of the excluded middle becomes true for all propositions whose negations make sense. "

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As to mathematics, it follow that the solution of a problem must be considered as an independent task (along with the proof of theoretical proposition).

Calculus of Problems and Propositions

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Cf. Melikhov's QHC (2022)

QHC

- given *proposition* p, formula $!p$ denotes the *problem* “to prove p”;
- given *problem* α, formula $?\alpha$ denotes the *proposition* “α has a solution”.

Notice that $?!p$ says “the problem of proving proposition p has a solution”, i.e., “p is provable”, and $!?\alpha$ says “to prove that problem α has a solution”, i.e., “to prove that α is solvable”.

QHC, contd

- $?!p \to p$ ("p is provable" implies p);
- $\alpha \to !?\alpha$ (solution of problem α solves the problem "to prove that α" is solvable");
- $!(p \to q) \to (!p \to !q)$ (a reduction of proof of q to proof of p reduces to proving implication $(p \to q)$);
- $?(\alpha \to \beta) \to (?\alpha \to ?\beta)$ (if the problem of reducing β to α is solvable then the solvability of α implies the solvability of β);
- $\neg !\bot$ (the falsity has no proof);

QHC, contd

and two additional rules of inference applying across QC and QH:

$$\frac{p}{!p} \text{ (CH)}$$

$$\frac{\alpha}{?\alpha} \text{ (HC)}$$

the intended interpretations of the rules

(CH): There is a method to find, for each formula F, a method deriving from a proof of the assertion that each of the propositions instantiating F is true a solution of the problem of proving (by a general method) all propositions instantiating F.

(HC): There is a method to find, for each formula Φ, a method deriving from any solution (by a general method) of all problems instantiating Φ a proof of the assertion that each of the problems instantiating Φ has a solution.

Galois connection between the Lindenbaum posets of equivalence classes QH-formuals and QC-formulas provided by ? and ! operators:

$$?\alpha \rightarrow p \text{ if and only if } \alpha \rightarrow !p$$

Conclusion 1

Kolmogorov's idea to distinguish between problems and propositions (theorems), and build a unified logical framework for both without dispensing with their differences, if considered against the intuitionist strategy to merge these things into one generic notion of proposition, receives an unexpected support from the homotopical interpretation of Martin-Löf's constructive type theory originally developed according to this strategy.

Conclusion 2

Kolmogorov's treatment of unsolvable Problems contains logical ideas that has been not fully realised yet in a formal setting, and thus can serve as a motivation of new research, in particular, in HoTT and related areas.

THANKS!