

The philosophy of Homotopy Type Theory (HoTT) examining its foundational implications for mathematics and logic

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Henri Poincaré Archives

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Plan :

General and Transcendental Logic according to Immanuel Kant

Philosophical Logic after Kant

Scholasticism in the Analytic Philosophy of the 20th century

Topology and Logic

HoTT against the Modern Philosophical Scholasticism

Conclusion

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Conclusion

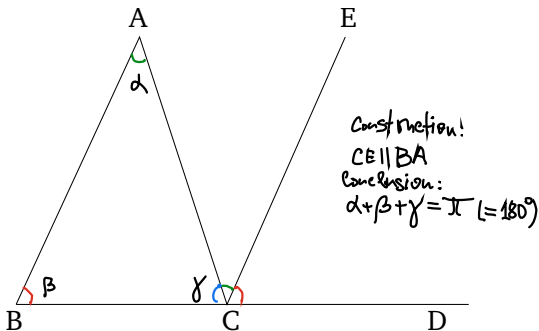
Kant on the Triangle Sum Theorem (1781)

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on the Triangle Sum Theorem (1781)

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (KrV, A 716 / B 744)

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- ▶ General (Formal) Logic : Logic of reasoning with bare abstract concepts : sufficient for philosophical/metaphysical speculation ;
- ▶ Transcendental Logic : Logic of reasoning about objects of possible experience, i.e., of mathematical reasoning that involves temporal (arithmetic) and spatial (geometry) intuitions : necessary for scientific reasoning (since objects of possible experience are always found in space/time).

Lobachevsky-Bolyai-Riemann : Non-Euclidean Geometry

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- ▶ Lobachevsky-Bolyai c. 1830

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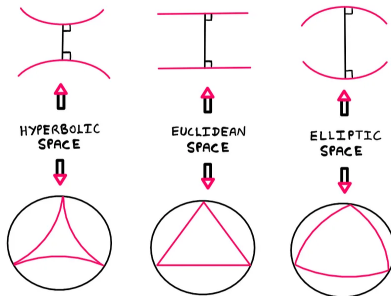
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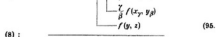
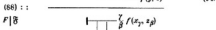
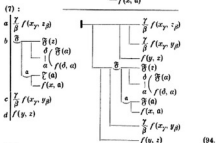
H. Weyl 1923 : The question of the validity of the “fifth postulate”, on which historical development started its attack on Euclid, seems to us nowadays to be a somewhat accidental point of departure. The knowledge that was necessary to take us beyond the Euclidean view was, in our opinion, revealed by Riemann.

Intrinsic Geometry of a Space



Frege : Non-Aristotelian Logic

Concept-Writing (*Begriffsschrift*) 1879.



Frege : Non-Aristotelian Logic

Function and Concept 1891

My starting-point is what is called a *function* in mathematics. [...] The first place where a scientific expression appears with a clear-cut meaning is where it is required for the statement of a law. This case arose as regards functions upon the discovery of higher Analysis [= Calculus]. Here for the first time it was a matter of setting forth laws holding for functions in general. So we must go back to the time when higher Analysis was discovered [i.e., the 17th century], if we want to know how the word 'function' was originally understood.

Russell 1903 : Mathematics is Logic

As I have attempted to prove in *The Principles of Mathematics*, when we analyse mathematics we bring it all back to logic. It all comes back to logic in the strictest and most formal sense.

Brouwer 1907 : Logic is an Auxiliary part of Mathematics

[M]athematical reasoning [...] is no logical reasoning [...] it uses the connectives of logic only because of the poverty of language, and thus may perhaps keep alive the language accompaniment even after the human intellect has already long ago outgrown the logical argument itself.

The Received “Logic-First” View : Frege, Hilbert (1899-1928-1934) - Tarski (1936)

Traditional “scholastic” setting : **Logic is First**. ZFC as a foundation of mathematics

“There is a single theory that starts from scratch : mathematical logic [...]. All other theories presuppose at least logic and usually a lot more. More precisely, the least a mathematical or a scientific theory takes for granted is the so-called ordinary [i.e., Classical] (two-valued) predicate calculus enriched with the microtheory of identity. [...] Every statement in mathematics or in science is, as far as its form is concerned, a formula of that calculus; and every valid reasoning is an instance of an inference pattern consecrated by that same theory.” (Mario Bunge 1972)

The Received “Logic-First” View : Interpretation of Axioms

Assumes a rigid distinction between logical and extra-logical (non-logical) terms and axioms of a theory. While the extra-logical elements admit for various interpretations (models) the semantic of logical elements is fixed and constitutes a universal formal framework where all reasoning is supposed to evolve.

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Thus in this framework the logical form of reasoning is rigidly fixed *semantically*.

Old and New Scholasticism

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The Emergence of First-Order Logic How did first-order logic come to be regarded as a privileged logical system — that is, as (in some sense) the “correct” logic for investigations in foundations of mathematics? That question, too, is highly complicated. Even after the Gödel results were widely understood, logicians continued to work in type theory, and it took years before first-order logic attained canonical status. The transition was gradual, and cannot be given a specific date. (SEP William Ewald 2018)

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Kantian Methodological Principle : Philosophical theses are not proved or disproved by (meta-)mathematical reasoning just like mathematical conjectures are not proved or disproved by philosophical arguments.

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- ▶ Intuitionistic Type theory (Martin-Löf since 1972) : Logic of Evidence

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- ▶ Homotopy Type theory.

Topology and Logic : Topological Semantics

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This alone could suggest a Kantian interpretation : Logic involves spatial concepts at the fundamental level. But it didn't. A contrast with Tarski's *Introduction to the Logic and to the Methodology of Deductive Sciences* (1936-1941).

Topology and Logic : Topos Logic

Lawvere since 1970 : Topos Logic

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Quantifiers and Sheaves 1970 : “The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. [...] A Grothendieck “topology” appears most naturally as a modal operator[...], the usual logical operators, such as \forall , \exists , \Rightarrow have natural analogues which apply to families of geometrical objects rather than to propositional functions.”

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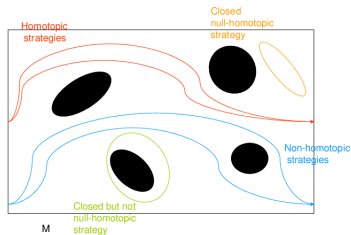
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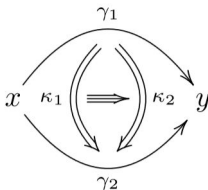
Interpretation of MLTT in terms of Homotopy theory : Types are spaces and terms are points. Logic is (a part of) Geometry !

Homotopy theory

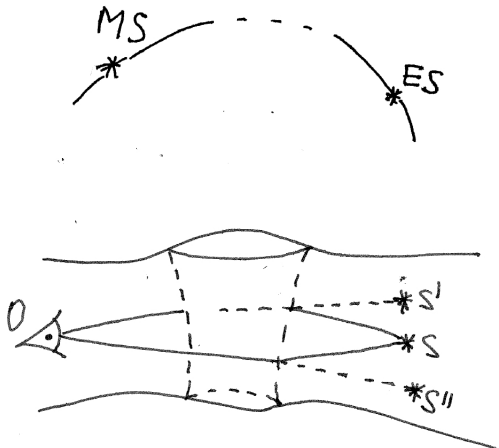


WARNING : a single curve at the above diagram represents many different *paths*. Unlike the concept of curve that of path is spatio-*temporal* rather than just spatial.

Higher Homotopies and Higher Groupoids



Identity types in HoTT



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Since HoTT involves the intuitive spatio-temporal reasoning at the fundamental level, HoTT qualifies as “transcendental” in Kant’s sense.

(This is assuming that spatio-temporal reasoning is a reasoning about objects of possible experience.) More precisely, HoTT can be described as a piece of intuitive mathematics where its transcendental logical part is made formal and explicit.

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It goes without saying that the human intuition is not fixed once and for all (biologically or otherwise) but evolves with the progress in mathematics, in the **visual arts** and in **music**.

David Corfield 2020 : a “New Logic for Philosophy” ? (once again after the passed century)

Yes, but we should avoid the idea of developing anything like “the” basic Logic as a replacement for the “Aristotelian” logic and for the “Fregean” logic in form of a particular logical calculus or a class of such calculi.

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Instead, we need to build and explore various system of Logic using both novel mathematical approaches and historically informed philosophical thinking expressed in natural languages.

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Directed Homotopy Type theory, which is presently a work in progress, suggests a further revision of foundations of mathematics and a stronger emphasis on the temporal aspects of logical reasoning.

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Directed Homotopy Type theory, which is presently a work in progress, suggests a further revision of foundations of mathematics and a stronger emphasis on the temporal aspects of logical reasoning.

See my *Does Identity Have a Sense?* forthcoming in *Manuscripto*.

Philosophy needs to pursue its critical revision of fundamental principles of reasoning using HoTT but it should avoid, by all means, making HoTT or any other advanced logical calculus into a new orthodoxy.

Thank You!