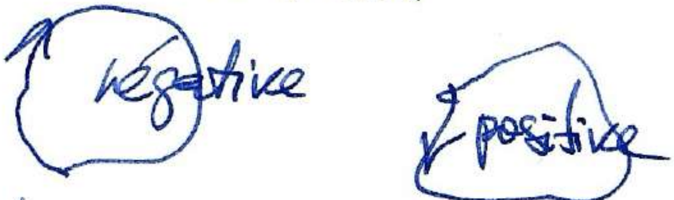
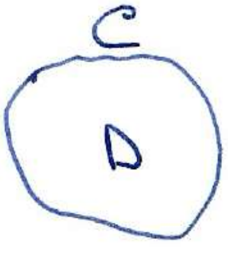
~~Green~~ Théorème de Green-Ostrogradski

la relation entre l'intégrale selon une courbe C (simple fermée) et l'intégral double sur la région D bornée par C

orientation



Si P, Q ~~ont~~ ses dérivées part. sur D, alors

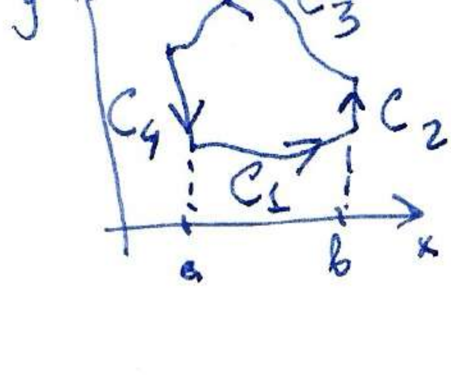


$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS$$

<u>Preuve</u> On montre que

(cas $D=\left\{(x,y) \,\middle|\, \begin{array}{c} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array}\right\}$) $\quad \displaystyle\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dS$ et

$$\int_C Q\,dy = \iint \frac{\partial Q}{\partial x}\,dS$$



$$\iint_D \frac{\partial P}{\partial y}\,dS = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx =$$

$$= \int_a^b \big(P(x,g_2(x)) - P(x,g_1(x))\big)\,dx \quad (*)$$

$$\int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

$$\int_{C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

$$\int_{C_2} P(x,y)\,dx = \int_{C_4} P(x,y)\,dx = 0 \quad \left(x = \text{const}, \ dx = 0\right)$$

$$\int_C P(x,y)\,dx = \int_{C_1} P\,dx + \int_{C_3} P\,dx = * \times (-1)$$

$$= -\iint_D \frac{\partial P}{\partial y}\,dS$$

Rotationnel $n=3$

$$\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$$

Def.: $$\text{rot}\,\bar{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\bar{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Mnémonique

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Le produit vectorielle

$$\underline{\vec{\nabla}\times\vec{F}} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \underline{\text{rot}\,\bar{F}}$$

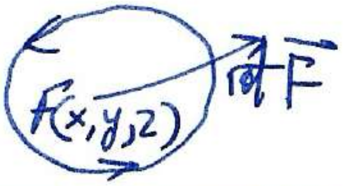
Th. Si $f(x,y,z)$ a ses der. partielles, alors $\text{rot}(\nabla f) = \vec{0}$

Preuve

$$\text{rot}(\nabla f) = \nabla\times\nabla f = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix} = +\left(\cancel{\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}}\right)\bar{i} + \ldots\ldots\bar{j} + \ldots\ldots\bar{k} = \vec{0}$$

$\rightarrow 0$ (Clairaut)

$\vec{F}$ conservatif ($\vec{F} = \nabla f$)



Cons. : Si $\vec{F}$ est conservatif, alors $\text{rot}\,\vec{F} = \vec{0}$

Divergence, $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

Déf. $$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ (scalaire)

$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$ la produit scalaire

Th. $\text{div}\,\text{rot}\,\vec{F} = 0$

Preuve $$\nabla \cdot \nabla \times \vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \cdots$$
$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \cancel{\frac{\partial^2 R}{\partial y \partial x}} + \cdots$$

$\nabla \vec{F} = 0$: flux incompressible

$$\text{div}\,\nabla f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

l'opérateur de Laplace

Les forme véctorielle de la th. de Green-Ost.

(1)

D

C

$$\int_C \overline{F}\cdot d\overline{r} = \int_C P\,dx + Q\,dy$$

$$\operatorname{rot} \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overline{k}$$

$$\operatorname{rot} \overline{F} \bullet \overline{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\boxed{\int_C \overline{F}\cdot d\overline{r} = \iint_D \operatorname{rot} \overline{F}\cdot \overline{k}\, dS}$$

(2) $\overline{r}(t) = x(t)\overline{i} + y(t)\overline{j} \qquad a \leq t \leq b$

vecteur tangent: $\overline{T}(t) = \dfrac{x'(t)}{\|\overline{r}'(t)\|}\overrightarrow{i} + \dfrac{y'(t)}{\|\overline{r}'(t)\|}\overrightarrow{j}$

vec. normal

$$\overrightarrow{n}(t) = \frac{y'(t)}{\|\overline{r}'(t)\|}\overline{i} - \frac{x'(t)}{\|\overline{r}'(t)\|}\overrightarrow{j}$$

$$\int_C \overline{F}\cdot\overline{n}\,dl = \int_a^b (\overline{F}\cdot\overline{n})(t)\|r'(t)\|\,dt =$$

$$= \int_a^b \left[\frac{P(x(t)), y(t))\, y'(t)}{\cancel{\|\bar{r}'(t)\|}} - \frac{Q(x(t), y(t))\, x'(t)}{\cancel{\|\bar{r}'(t)\|}} \right] \times$$

$$\times \cancel{\|r'(t)\|}\, dt =$$

$$= \int_a^b \left[P(x(t), y(t))\, y'(t) - Q(x(t), y(t))\, x'(t) \right] dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dS$$

$$\boxed{\int_C \bar{F} \cdot \bar{n}\, dl = \iint_D \operatorname{div} \bar{F}\, dS}$$