Понятие математической структуры согласно Владимиру Воеводскому

Андрей Родин (Archives Henri Poincaré, université de Lorraine, France)

ВШЭ Москва, семинар “Математическия логика и теория категорий”

8 января 2025

Владимир Воеводский (1966-2017)

Этот доклад посвящен памяти Владимира Воеводского

Plan:

Fields Medal of 2002

(From Notices of the AMS, vol. 49, no 10, by Allyn Jackson)

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A major advance came when Voevodsky, building on a little-understood idea proposed by Andrei Suslin, created a theory of "motivic cohomology".

Fields Medal of 2002 (from Notices of the AMS, vol. 49, no 10)

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[In 2009 Voevodsky proves the Bloch-Kato Conjecture that generalises the John Milnor's Conjecture]

Interview with Roman Mikhailov, 2012

Начиная с осени 1997 я уже понимал, что мой основной вклад в теорию мотивов и мотивные когомологии сделан. С этого времени я очень осознано и активно искал тему которой я буду заниматься, когда выполню свои обязательства, связанные с гипотезой Блоха-Като. [. . .] Видя тенденции развития математики как науки, я понимал что [. . .] математика находится на пороге кризиса, а точнее двух кризисов.

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Первый связан с отрывом математики "чистой"от математики прикладной. Понятно, что рано или поздно встанет вопрос о том, а почему общество должно платить деньги людям, которые занимаются вещами, не имеющими никаких практических приложений.

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Второй, менее очевидный, связан с усложнением чистой математики, которое ведет к тому, что, опять же рано или поздно, статьи станут слишком сложными для детальной проверки и начнется процесс накопления незамеченных ошибок.

Interview with Roman Mikhailov, 2012, cntd

Значит, решил я, нужно попытаться сделать нечто, что поможет предотвратить эти кризисы. В первом случае это означало, что нужно найти такую прикладную задачу, которая бы требовала для своего решения методов чистой математики, разработанных в последние годы или хотя бы десятилетия. [. . .]

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В результате я выбрал, как сейчас понимаю неправильно, проблему восстановления истории популяций по их современной генетической композиции. Я провозился с этой задачей в общей сложности около двух лет и в конце концов, уже в 2009 году, понял, что то, что я придумывал, бесполезно.

Vladimir's unfinished project, see https://arxiv.org/abs/2012.01150

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Voevodsky's unfinished project: Filling the gap between pure and applied mathematics

Andrei Rodin

Institute of Philosophy, Russian Academy of Sciences and Saint-Petersburg State University, Russia

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ABSTRACT

In a series of lectures given in 2003, soon after receiving the Fields Medal for his results in the Algebraic Geometry, Vladimir Voevodsky (1966–2017) identifies two strategic goals for mathematics, which he plans to pursue in his further research. The first goal is to develop a "computerised library of mathematical knowledge," which supports an automated proof-verification. The second goal is to "bridge pure and applied mathematics." Voevodsky's research towards the first goal brought about the new Univalent foundations of mathematics. In view of the second goal Voevodsky in 2004 started to develop a mathematical theory of Population Dynamics, which involved the Categorical Probability theory. This latter project did not bring published results and was abandoned by Voevodsky in 2009 when he decided to focus his efforts on the Univalent foundations and closely related topics. In the present paper, which is based on Voevodsky's archival sources, I present Voevodsky's views of mathematics and its relationships with natural sciences, critically discuss these views, and suggest how Voevodsky's ideas and approaches in the applied mathematics can be further developed and pursued. A special attention is given to Voevodsky's original strategy to bridge the persisting gap between pure and applied mathematics where computers and the computer-assisted mathematics play a major role.

Interview with Roman Mikhailov, 2012, cntd

Параллельно я искал подходы к проблеме накопления ошибок в работах по чистой математике. Было ясно, что единственное решение - это создание языка, на котором математические доказательства могут писаться людьми в такой форме, что это можно будет проверять на компьютере.

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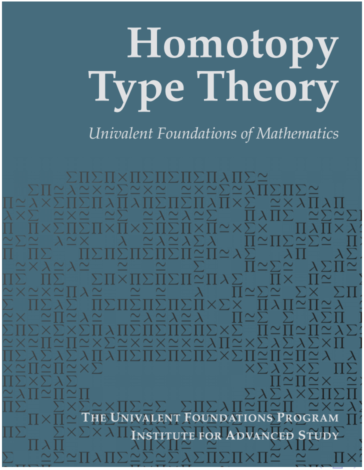
Вплоть до 2005 года мне казалось, что это задача намного более сложная чем задача исторической генетики, которой я занимался. [... Но в] 2005 году мне удалось сформулировать несколько идей, которые неожиданно открыли новый подход к одной из основных проблем в основаниях современной математики.

HoTT Book 2013: an outcome of the Special Year (2012-2013) on Univalent Foundations organised by IAS Princeton



HoTT Book authors

Preface

IAS Special Year on Univalent Foundations

A Special Year on Univalent Foundations of Mathematics was held in 2012-13 at the Institute for Advanced Study, School of Mathematics, organized by Steve Awodey, Thierry Coquand, and Vladimir Voevodsky. The following people were the official participants.

Peter Aczel	Eric Finster	Alvaro Pelayo
Benedikt Ahrens	Daniel Grayson	Andrew Polonsky
Thorsten Altenkirch	Hugo Herbelin	Michael Shulman
Steve Awodey	André Joyal	Matthieu Sozeau
Bruno Barras	Dan Licata	Bas Spitters
Andrej Bauer	Peter Lumsdaine	Benno van den Berg
Yves Bertot	Assia Mahboubi	Vladimir Voevodsky
Marc Bezem	Per Martin-Löf	Michael Warren
Thierry Coquand	Sergey Melikhov	Noam Zeilberger

There were also the following students, whose participation was no less valuable.

Carlo Angiuli	Guillaume Brunerie	Egbert Rijke
Anthony Bordg	Chris Kapulkin	Kristina Sojakova

In addition, there were the following short- and long-term visitors, including student visitors, whose contributions to the Special Year were also essential.

Jeremy Avigad	Richard Garner	Nuo Li
Cyril Cohen	Georges Gonthier	Zhaohui Luo
Robert Constable	Thomas Hales	Michael Nahas
Pierre-Louis Curien	Robert Harper	Erik Palmgren
Peter Dybjer	Martin Hofmann	Emily Riehl
Martín Escardó	Pieter Hofstra	Dana Scott
Kuen-Bang Hou	Joachim Kock	Philip Scott

Vladimir's vision of mathematics

During my visit to IAS in February 2015 Vladimir told me that he did not take part in the collective work on the HoTT Book. (This is confirmed by other participants of the Special Year.) Vladimir did not approve on the Book and considered to formally withdraw his name from the list of its authors.

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The present paper makes part of my effort to reconstruct Vladimir's vision, understand and critically assess it, and possibly further develop.

Automated poof verification and proof search

The idea of automated proof (including proof verification and proof search) emerged in the 1950s along with the idea of Automated Intelligence. The official date of birth of automated proof is December 1968 when a special conference on the topic was held in Versaille. Working software AUTOMATH was presented at that conference by Nicolas de Bruijn.

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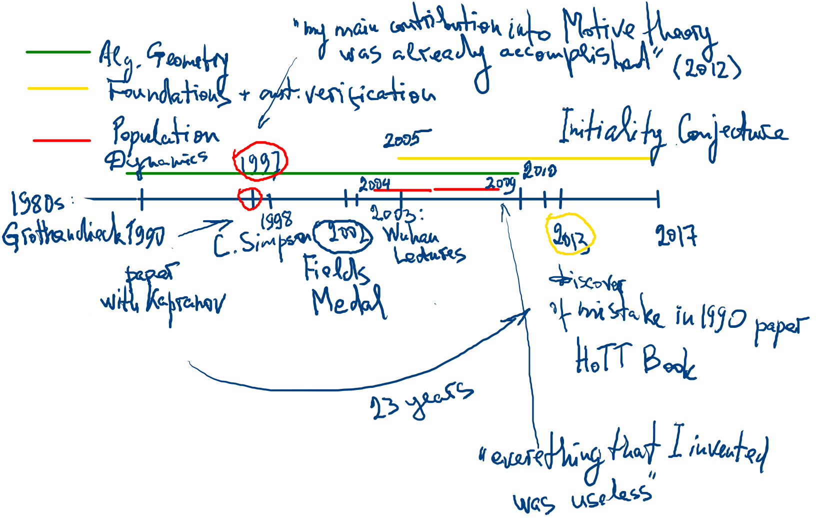
In the USSR researches in automated proof were pursued since 1962 in the newly created Institute of Cybernetics of Ukrainian Academy of Science in Kiev under the supervision and directorship of Viktor Mikhailovich. Glushkov (Виктор Михайлович Глушков). In 1970 Glushkov published a fundamental paper (Неторые проблемы теории автоматов и искусственного интеллекта), in which he formulated the modern concept of interactive proof-assistant and suggested the idea of "evidence algorithm" (алгоритм очевидности), which was supposed to support a growing library of formalised mathematics. Glushkov and his team successfully implemented these ideas in the so-called "system of automated proof" (система автоматического доказательств).

Automated poof verification and proof search

Today there are many proof assistants on the market (Coq, Isbell, Nurpl, more recently Lean), which do NOT use HoTT/UL

Timeline



"my main contribution into Motive theory
was already accomplished" (2012)
Alg. Geometry
Foundations + aut. verification
Population Dynamics
Initiality Conjecture
2005
1997
2010
2004
2009
1980s:
Grothendieck 1990
1998
C. Simpson
2002
Fields Medal
2003:
Wuhan Lectures
2013
2017
paper with Kapranov
discovery of mistake in 1990 paper
HoTT Book
23 years
"everything that I invented
was useless"

Voevodsky to Rodin, January 27, 2016

У Прокла есть четкое понимание различия между свойством и структурой (см. его рассуждение о том что есть угол в комментарии на Евклида). Отношение это совместное свойство двух или более объектов. В философии и математической логике основой модели ¥мира¥ является совокупность предметов и совокупность отношений между ними т.е. совокупность свойств наборов предметов.
ПС Отношение между предметами или есть или его нет (две прямые или параллельны или нет), а структура может иметь больше одного представителя (см. Евклида Прокла 123 по стандартной нумерации).

Voevodsky to Rodin, January 27, 2016, contd

Вопрос который меня интересует это когда и как возникла и распространилась странная идея что основу мира можно описать совокупностью отношениями между предметами в противоположность совокупности совместных структур на наборах предметов.

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когда эта идея была явно представлена как подсобное упрощение - давайте мол попробуем для начала такой до смешного упрощенный вариант как в математике рассматривают упрощенный вариант задачи чтобы опробовать на нем ту или иную общую идею.

Volodya

Voevodsky to Rodin, January 27, 2016, contd

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Rodin to Voevodsky Janaury 30, 2016

Такое понимание отношения в истории логики очень позднее; точную формулировку такого рода впервые можно найти только у Фреге (хотя мб он что-то и использовал из более раннего и сегодня забытого, это надо проверять). Рассел считал, что это изобретение Фреге позволяет обойти все возражения Канта по поводу возможности сведения математики к логике [. . .]

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Russell 1918: "As I have attempted to prove in *The Principles of Mathematics*, when we analyse mathematics we bring it all back to logic. . . . In the present lectures, I shall try to set forth in a sort of outline, rather briefly and rather unsatisfactorily, a kind of logical doctrine which seems to me to result from the philosophy of mathematics — not exactly logically, but as what emerges as one reflects: a certain kind of logical doctrine, and on the basis of this a certain kind of metaphysics."

Rodin to Voevodsky Janaury 30, 2016, contd.

А ты мог бы уточнить, как именно ты понимаешь структуру? Если это не множество с набором отношений, то что?

Andrei

Voevodsky to Rodin, 31, 2016, contd

В контексте который был в моем сообщении структура на нескольких объектах это нечто что может связывать эти объекты несколькими разными способами. Отношение оно или есть или его нет, т.е., здесь связь или есть или нет, множество возможных вариантов этой связи есть или пустое множество или множество из одного элемента. Структура это связь множество вариантов которой может иметь больше одного элемента. Заметь, что при таком определении отношение между А и Б есть частный случай структуры на совокупности {А,Б}.

Voevodsky to Rodin, 31, 2016, contd

Что касается того что идея использовать отношения появилась сначала в логике то я в этом сомневаюсь. Посмотри “Метафизику” Лотзе. Там, первые страниц сто посвящены обсуждению картины мира в которой основой является множество объектов и отношений между ними. Хотя это по моему после Фреге (кстати в первой работе Буля, тоже объекты и отношения, кстати Буль может быть даже дает ссылку на то откуда он это взял), я сомневаюсь чтобы Лодзе читал Фреге (или Буля).

Volodya

Euclid's Def. 1.8.

Плоский угол — это наклон друг к другу двух линий, которые в плоскости встречают друг друга и не лежат на одной прямой.

Remarks on Euclid's Def.1.8.

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- Def. 1.8. defines a plane linear angle, which covers curvilinear angles (e.g. between a tangent to a circle and the circumference of this circle). The concept of rectilinear plane angle is defined in the following Def.1.9.: Когда угол заключают прямые линии [то есть стороны угла прямые], он называется прямолинейным.

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- In Book 11 of the *Elements*, which belongs to the stereometric part of the *Elements*, one finds more angle-related definitions (Def.11.5,6,7) including the definition of *solid* (i.e. 3-dimensional) angle (Def.11.11).
- One does not find among these definitions, however, the concept of two-dimensional non-plane angle like an angle formed by two intersecting great circles of a sphere.

Proclus' problem

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- relation (Def.1.8.)
- quality (a special kind of figure like a circle),
- quantity (a measurable magnitude)

Proclus' conclusion

Proclus' conclusion is uncertain: he claims that angles do not fall under any of the three categories precisely but at the same time combine aspects of all these categories.

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In the relevant fragment of his *Commentary* Proclus (1) assumes that Euclid's Def.1.8. defines angle as a relation (of "inclination" (взаимный наклон) of two intersecting lines) and (2) provides an argument intended to show that this definition is not appropriate.

Proclus' commentary on Def.1.8.

Если угол является наклоном, [as in Euclid's Def. 1.8.] и в целом отношением, то тогда одному наклону соответствует один угол, а не многие. Ведь если угол - не что иное, как отношение линий или плоскостей, как может быть одно отношение, но много углов?

Proclus' commentary on Def.1.8.

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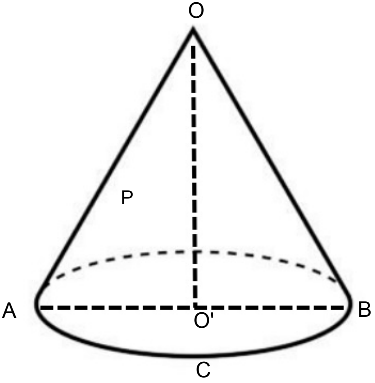
Если ты представишь себе конус, рассеченный треугольником от вершины до основания, ты увидишь при вершине полуконуса один наклон линий треугольника, но два разных угла: один — плоский угол треугольника, другой — угол на смешанной поверхности конуса; однако оба эти угла ограничены двумя упомянутыми линиями. Стало быть, отношение линий не производит угла.

Proclus' cone construction

The same relation of inclination between straight lines OA, OB produces two different angles $\angle AOB_P$ and $\angle AOB_C$.



Proclus' argument in Vladimir' terms

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Angles $\angle AOB_P$ and $\angle AOB_C$ link lines OA, OB in two different ways.

HoTT

In HoTT/UF we have a similar situation: terms (points) A, B are identified via different non-homotopical paths p, q; homotopical paths can be identified via non-equivalent path homotopies, and so on up to the infinite hierarchy of homotopy levels.

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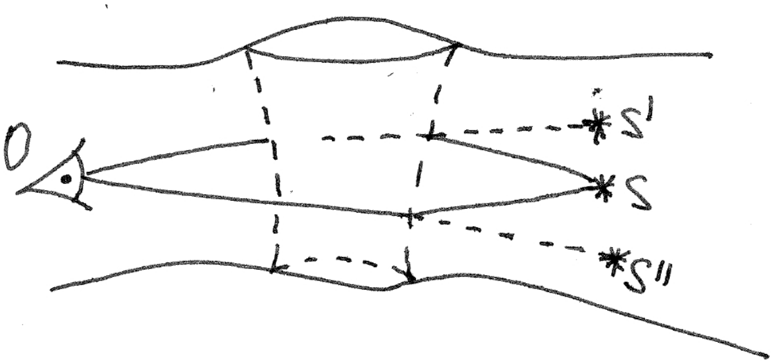
Example: higher identity types : $q, p : A =_S B$, $h_1, h_2 : q =_{A=_S B} p$, ...

Identity is a multi-level structure but not a mere relation! In HoTT this feature of identity predicate is transferred ("transported") to all predicates (including non-monadic ones usually called and understood as relations).

Example: hypothetical gravitational lensing around a wormhole in spacetime



Is Proclus' cone construction relevant to Def.1.8.?

Proclus' argument involves a non-plane angle $\angle AOB_C$. Such angles are not defined in Euclid's *Elements*. One may argue that this makes Proclus' argument irrelevant.

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Proclus' argument can be saved, nevertheless, with the following additional argument: since (two-dimensional non-plane) angle cannot be defined as a relation of lines in the solid (3D) geometry, it should not be so defined in the plane (2D) geometry either — because the plane geometry is a special case of the solid geometry

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What is wrong then about the fact that in Proclus' cone construction two (in fact, three!) rather than just one angle fall under Euclid's definition?

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An instance of a binary relation, (say $OA \bowtie OB$ where $\bowtie$ stands for "inclination"), is fully identified when its relata (OA and OB in our case) are identified.

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Corollary:

An instance of a binary relation is an (ordered) pair of items that hold that relation.

Grounding **RU**: Pairing in ZF

For all items (sets) X, Y, there exist *unique* non-ordered pair (X, Y). (Pairing and Extensionality in the ZF). This suffices for symmetric relations like $\bowtie$.

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For non-symmetric relations one needs a notion of ordered pair. The similar uniqueness principle for ordered pairs can be justified with ZF only via fixing a particular representation of ordered pairs with sets like

- $< X, Y >:= \{\{X\}, \{X, Y\}\}$ (Kuratowski)
- $< X, Y >:= \{\{X, 1\}, \{Y, 2\}\}$ (Haussdorf)
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The identification of ordered pairs across their set-theoretic representations is a Benacerraf-style challenge.

Proclus' argument in the nutshell (classically)

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3. Using the cone construction show that pair (OA, OB) such that $OA \bowtie OB$ gives rise to two different angles $\angle AOB_P$ and $\angle AOB_C$.
4. Conclude that angle cannot be soundly defined as pair lines such as (OA, OB) that stand in a certain relation (such as Euclid's relation of inclination $OA \bowtie OB$).

Is the above reconstruction of Proclus' argument illuminating?

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In what follows I provide an alternative *constructive* reconstruction of Proclus' argument, which is based on constructive logical principles used in HoTT.

The truth of a proposition and the validity of a judgement (after Martin-Löf)

$OA \bowtie OB$ is a *proposition*, $\angle AOB : OA \bowtie OB$ is a *judgement* where $\angle AOB$ *witnesses* (proves) the fact that relation $OA \bowtie OB$ holds (and hence proposition $OA \bowtie OB$ is true).

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In the constructive logic one does not assume that propositions have their truth-values independently of one's reasoning. A proposition qualifies as *true* if it has a proof. It qualifies as *false* if there is a proof that it has no proof. A proposition may remain undecided and/or be provably undecidable.

Constructive version of **RU**

RUC:

Proposition P (in particular, one of the form xRy) has *at most one* proof (aka witness).

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4. Using the cone construction show that the pair (OA, OB) gives rise to two different angles $\angle AOB_P$ and $\angle AOB_C$ none of which can be identified with the hypothetical angle $\angle AOB$.
5. Conclude using **RUC** that angles cannot be defined (and identified) as *the* witnesses of a proposition of the form $X \bowtie Y$. Thus angle cannot be defined as a relation (between its sides).

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In order to reply to this straightforward objection, one needs to be more specific about the concept of proposition. In HoTT there is a convention according to wich only types with at most one term are called by the name of proposition (or *mere* proposition).

Homotopical hierarchy of types

The idea behind this terminological choice is that proposition P with set S is a higher-order *structure* that should be distinguished from the proposition itself. P can be obtained from S by identifying all its elements if there are some, and leaving it empty otherwise. So we get constructive counterparts of classical truth-values: a proposition either has a proof (i.e., is constructively true), or (provably) has no proof (i.e., is constructively false).

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Then come *2-groupoids* interpreted as above plus distinguishable homotopies (up to 2-homotopies), and so on up along the homotopical ladder.

Following Proclus: Is (group) isomorphism a relation?

No, because given a pair of groups G_1, G_2 one can, generally, identify more than one isomorphism of the form $i : G_1 \xrightarrow{\sim} G_2$.

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The concept of iso-relation is a *truncated version* of that of iso-map (or iso-cell). In this case the truncation amounts to collapsing of all iso-maps (iso-cells) of a given type into a single term (object).

Following Proclus: Is (group) isomorphism a relation?

Now let us consider the same problem in a constructive setting. Suppose we have an algorithm A that for a given pair of groups G_1, G_2 either effectively computes an iso-map $i : G_1 \xrightarrow{\sim} G_2$ or proves that such an iso-map does not exist. Thus A decides whether $G_1 \sim G_2$ holds by constructing an iso-map. Does judgement $i : G_1 \xrightarrow{\sim} G_2$ fully convey us the "constructive meaning" of isomorphism?

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Remark: The isomorphism problem is known to be algorithmically *undecidable* in the general case. (The decidability problem was formulated by M. Dehn in 1911 and solved by P.S. Novikov in 1955). The group isomorphism problem is, however, decidable for some special classes of groups.

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Since judgment $i : G_1 \sim G_2$ requires performing iso-map i, knowing that $G_1 \xrightarrow{\sim} G_2$ and knowing-how to perform i amount to the same. So one may be inclined to answer the above question positively.

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Cf. Quine's motto *No entity without identity*. The type of iso-maps of the form $i : G_1 \sim G_2$ is a set-level type but not a propositional type .

Structuralist Principle

Mathematical Structuralism is a view according to which mathematical objects are *structures*. This view exists in many varieties that provide more than one conception of mathematical structure. But all these conceptions share the following basic structuralist principle (**SP**):

Isomorphic structures are the same.

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Mathematical Structuralism as presented in Jean Dieudonné's manifesto of 1950 *The Architecture of Mathematics* was a driving idea behind Bourbaki's project of (re-)building mathematics on set-theoretic foundations.

Bencarraf Problem and the "Disappearance of sets"

But as it was stressed by Paul Bencerraf in 1965, the standard set-theoretic foundations could not support **SP**: two sets that represents isomorphic structures generally, are not equal (in the sense of the equality predicate making part of the underlying set theory). Think of natural numbers, ordered pairs, etc

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“[T]he notion of set itself disappeared [. . .] in the light of the recent work on logical formalism”, so that “mathematical structures become, properly speaking, the only “objects” of mathematics.”

As far as I can see, this claim in 1950 was nothing but a wishful thinking.

SP and Univalence

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This is how the Univalence Axiom can be put in words (with some risk of misinterpreting the mathematical statement):

$$e \ : (A =_{TYPE} B) \simeq (A \simeq B)$$

Vladimir's conception of mathematical structure vs. the standard conception

In some of his talks Vladimir mentions **SP** as a motivation for developing UF but he does not include it in his 2014 programmatic talk and paper *The Origins and Motivations of Univalent Foundations*. Apparently he considers the solution of the Benacerraf problem as a secondary issue.

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Remarkably, in HoTT/UF the two conceptions of structure have the same *extension*: each type is a structure in the standard sense (as an object satisfying **SP**) and each type is a structure in Vladimir's sense. But Vladimir's conception of structure unlike the standard conception (in the given setting) does not require the Univalence.

Mathematics beyond the set level

Vladimir's motivation behind his conception of structure was to develop new mathematics beyond the Borubaki-style set-level mathematics. As an example of such mathematics Vladimir could point to Motive theory.

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Motive theory could be and was developed earlier without the HoTT/UF machinery but this machinery, in Vladimir's view, could provide a better formal foundation for it and support automated proof-checking of proofs in this theory. The project of UF-formalising and proof-checking of Vladimir's results in Motive theory has not been realised during his lifetime and remains so far an open problem.

Structures of higher homotopy levels often (perhaps always?) can be encoded into the standard set-level structures. For example a groupoid can be so encoded as a multigraph with a partical operation of composition on its arrows. But such a reduction of higher homotopy levels to the level of sets is not epistemically advantageous and thus does not devaluate the idea of "going beyond" the set-level mathematics.

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- 2013: Solution of Benacerraf Problem with UF (accomplishment).

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Does Structuralism have a future?

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Analysing UF through the optics of the old-fashioned 20th century Mathematical Structuralism is not sufficient, in my view, for explaining its epistemological significance.

Today — as ever — philosophers need to be as attentive and as receptive to novel developments in mathematics as were Cassirer and Russell in the 1900s. The Higher-Dimensional Algebra and the Higher Category theory evidently do bring along such novel ways of reasoning, which are quite unlike those codified in the standard Structuralist doctrines, which date back to the early 20th century.

Categories without Structures

In my *Categories without Structures* of 2011 (*Philosophia Mathematica, vol. 19, no. 1*) I objected the popular view according to which Category theory supports the Mathematical Structuralism and sketched an alternative interpretation of CT that emphasised novel features of this theory non reducible to the Hilbert-style and Bourbaki-style ways of mathematical reasoning and of organising mathematics.

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A key point of my critique: isomorphism is a map but not a relation, so the **SP** does not adequately capture the working of this concept in CT. Another point of my critique (earlier stressed by Grothendieck): general morphisms are as important in CT and CT-based mathematics as are isomorphisms.

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I believe that the notion of "Category-Theoretic version of Mathematical Structuralism" defended by a number of philosophers is an attempt to put a new wine in old bottles.

Philosophy of HoTT

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Vladimir's unconventional conception of mathematical structure reflects a novel way of mathematical reasoning emerged on the threshold the 21st century and needs a close philosophical analysis.

Does Vladimir's historical question make sense?

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"I doubt that the idea to use relations [instead of general structures] first appeared in logic. Look at [Hermann] Lotze's Metaphysics [1841]. [. . .] . By the way, in the first Boole's work [*The Mathematical Analysis of Logic* of 1847] we also find objects and relations. Perhaps Boole provides a reference pointing to his source. But I doubt that Lotze read Frege or Boole."

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5. Vladimir's distinction between relations and (higher order) structures provides a novel conceptual optics for studying and interpreting the history of relation concept from Aristotle onwards.

The present paper under the title "Vladimir Voevodsky on the concept of mathematical structure in his letter exchange with Andrei Rodin" is submitted to Springer (Synthese Library) volume *Mathematicians at Work: Empirically informed Philosophy of Mathematics* edited by Deborah Kant, José Antonio Perez-Escobar, Deniz Sarikaya, and Mira Sarikaya scheduled to appear in 2025+.

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Thank You!