The concept of mathematical structure according to Voevodsky

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NYC Category Theory Seminar, CUNY

April 23, 2025

Vladimir Voevodsky (1966-2017)

Plan:

1. Mathematical Structuralism and Univalent Foundations (the received narrative)
2. Correspondence
3. Proclus on Euclid's definition of plane angle
4. Some hints to Homotopy Type theory
5. Two reconstructions of Proclus' argument
   - Classical reconstruction
   - Constructive reconstruction
6. Conclusion: Structures and Structuralism beyond the Set-Level Mathematics

Structures

- Mathematical structures are usually conceived of as *models* of some (sets of) *axioms*.
- Structures are identified *up to isomorphism*: isomorphic structures count as the same. An appropriate isomorphism concept depends on (is fully determined by?) the axioms.
- Example: Groups. Axioms involve primitive individuals (elements of groups), and primitive relations (group operation as a ternary relation). A (single) group is a model of the axioms. The theory is not categorical: there exist non-isomorphic groups. The concept of group isomorphism involves the concepts of (i) *element* of a group and (ii) group *operation* introduced with the axioms.

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- Theory: Peano Arithmetic
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- Model 1: finite von Neumann ordinals: $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \dots$
- Model 2: Zermelo ordinals: $\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots$
- Are the two models isomorphic? Obviously.

But what is the isomorphism exactly?

$$i : \mathbb{NEU} \xrightarrow{\sim} \mathbb{ZER}$$

Hindu numerals:	1	2	3	...
$\mathbb{NEU}$	$\{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$	$\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$	...
$\mathbb{ZER}$	$\{\emptyset\}$	$\{\{\emptyset\}\}$	$\{\{\{\emptyset\}\}\}$	...

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i does not reflect any shared internal structure of Zermelo and von Neumann ordinals but reflects the fact that $\mathbb{ZER}$ and $\mathbb{NEU}$ are generated by recursive procedures that can be easily *synchronised*. So the only common feature of $\mathbb{ZER}$ and $\mathbb{NEU}$ is that both are presentable as *series*.

Any recursively generated series will do

Any infinite series (i.e., an effectively enumerated infinite set) is a (standard) model of **PA** on equal footing with $\mathbb{ZER}$ and $\mathbb{NEU}$.

Hindu numerals:	1	2	3	...
$\mathbb{NEU}$	$\{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$	$\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$	...
Notching numerals	$\vert$	$\vert\vert$	$\vert\vert\vert$	...
Any recursively defined series	...	...	...	...

Any recursively generated series will do

From: Caleb Everett, *Numbers and the Making of Us: counting and the course of human cultures*, Harvard University Press, 2017

2.1. The reindeer antler of Little Salt Spring, Florida, with a colleague's hand for scale. Photograph by the author.

The artefact has been found in the mid 2010s in Florida; it is c. 10K years old. The smaller notches in one-one correspondence to the larger ones is an evidence that the device was used for counting. The notches may represent the 29 days of the Lunar cycle.

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$\mathbb{ZER}, \mathbb{NEU}$ are popular representations of $\mathbb{N}$ only because $\emptyset$ is the only set the uniqueness and the existence of which is guaranteed by the ZF, so in the set-theoretic framework it can be thought of as "an atom out of which all sets are built". Indeed the above quasi-constructive procedure has been used by von Neumann for modelling the whole of **ZF**. (This can be a reason to prefer $\mathbb{NEU}$ as the stadard presentation of natural numbers)

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There is no common *intrinsic* property of sets $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ and $\{\{\{\emptyset\}\}\}$ that may justify the idea that both represent the same number 3. They both represent this number because they both are obtained at the 3d step of certain recursive procedures.

Problem

In Number theory one studies properties of *standard* models of PA, i.e., properties, which are shared by all models of PA isomorphic to $\mathbb{N}$. Let us call such properties *arithmetical*. Because of G1 this class of properties cannot be identified with those provable in PA.

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How such properties can be formally distinguished otherwise? Which set-theoretic properties of Zermelo ordinals and von Neumann ordinals qualify as arithmetical?

Group theory

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What is colloquially called 'Group theory' is not a study of consequences of those axioms but a study of their (usually set theoretic) *models*. The two notions of Group theory should be kept apart. Like in the case of PA and $\mathbb{N}$, theorems of what is colloquially called 'Group theory' (like Lagrange theorem), are not, generally, logical consequences of axioms of Group theory.

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Like in the case of Number theory, it is not immediately clear which expressible properties of groups (say, expressible in the language of ZF) are 'group-theoretic', i.e., invariant under isomorphisms, and which are not.

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In the set-based mathematics there is no satisfactory formal account for the idea of taking isomorphisms for identities. There is a systematic ambiguity of expressions like "the same group". In some contexts this may refer to the same group structure up to isomorphism, and in some other contexts – to particular "isomorphic copies" of the group.

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A similar wobble is characteristic for the standard (flat) Category-theory. Objects in a category are typically understood up to (unique) isomorphism while the equality of morphisms is supposed to be strict. The equality of categories is typically understood up to equivalence of categories (rather than up to isomorphism). Many people (Geoffrey Hellmann et al.) promoted a category-theoretic version of Mathematical Structuralism on that ground.

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Let G, H be two mathematical structures of the same type T (i.e. subject to the same axioms) and let G, H be isomorphic. Then for any *structural* property P it is the case that structure G has property P if and only if H has property P. In symbols:

$$\forall P. (G \sim H) \Rightarrow (P(G) \leftrightarrow P(H))$$

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The *indiscernibility of identicals* is a minimalistic requirement for one's identity concept. It is less controversial than the converse *identity of indiscernibles* principle. I discuss here only the former.

IEP fails in ZF

In ZF and ZF-based mathematics IEP is not supported. This is because it obviously fails for the general ZF notion of isomorphism (i.e., invertible function: not every such function is a group isomorphism), and there is no universal method in view that would allow one to effectively distinguish between structural and non-structural properties for a given structure type.

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In other words, given a theory T and the corresponding type of structures (like groups), there is no general method that might allow one to separate in ZF *structural* (i.e. isomorphism-invariant) properties of T-structures from their other properties.

UF: the structuralist dream comes true

But in HoTT/UF, IEP holds for a large classe (of categories of) structures which include all familiar set-based structures like groups, rings, topological spaces, etc. Precisely, IEP holds for *univalent* categories.

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UF restricted to univalent categories makes the old structuralist dream true.

UF and Structuralism

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Vladimir mentioned this feature of UF in a 2006 talk (before the UF got its current name) but interestingly it is missing in his later talks including his 2014 IAS lecture on his motivations behind his work on UF https://www.ias.edu/ideas/2014/voevodsky-origins. Apparently in Vladimir's view IEP was a secondary issue.

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As we shall now see Vladimir developed very different ideas about mathematical structures in the UF.

Voevodsky to Rodin, January 27, 2016

In Proclus we find a clear distinction between a property and a structure: see his reflection on how to define an angle in his Commentary on Euclid. Relation is a joint property of two or more objects. In philosophy and in mathematical logic the model of the "world" is based on a collection of objects and a collection of relations between these objects, i.e., a collection of properties of assemblies of objects.

Voevodsky to Rodin, January 27, 2016, contd

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There should be a [historical] moment when this idea was first presented as an auxiliary simplification : [as if someone says] “let’s assume this laughably simplified version” just like in mathematics people often consider a simplified version of a problem in order to test with it this or that general idea.

Volodya

Voevodsky to Rodin, January 27, 2016, contd

P.S. A relation either holds or doesn't hold (straight lines are either parallel or not) but a structure can have more than one representative.

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Proclus' *Commentary on Euclid* is a major source of our present knowledge of the history of Greek mathematics.

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I must confess that, I first did not take this historical question seriously and only later realised that Vladimir was as bright and profound in his historical studies as he was in mathematics. This is a topic for another talk.

Rodin to Voevodsky January 30, 2016

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Could you clarify what you call a structure?

If it is not a set with a collection of relations then what it is? A structure can be also understood as a set with relations [identified] up to isomorphism — in this case a structure can have more than one representative [that is, more than one instantiation]. Does this concept of structure fit to what you are talking about?

Andrei

Voevodsky to Rodin, January 31, 2016

[A] structure on several objects is an entity that can link these objects in a number of different ways. A relation [between two given objects] either holds or does not hold, i.e., either there is a link [between these objects] or not, so the set of possible versions of this link is either empty or has one element. A structure is a link such that the set of its possible versions can have more than one element.

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Notice that given this definition [of structure], a relation between A and B is a special case of structure on collection {A,B}.

Volodya

Comment 3:

At that point I (convinced myself that I) understood what Volodya was talking about. I shall now try to explain it to the audience.

Euclid's Def. 1.8.

[A] plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.

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- In Book 11 of the *Elements*, which belongs to the stereometric part of the *Elements*, one finds more angle-related definitions (Def.11.5,6,7) including the definition of *solid* (i.e. 3-dimensional) angle (Def.11.11).
- One does not find among these definitions, however, the concept of two-dimensional non-plane angle like an angle formed by two intersecting great circles of a sphere.

Proclus' problem

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To classify geometrical angles into (one or more of) these there categories borrowed from Aristotle's list of 10 categories found in his *Organon*:

- relation (Def.1.8.)
- quality (a special kind of figure like a circle),
- quantity (a measurable magnitude)

Proclus' conclusion

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In the relevant fragment of his *Commentary* Proclus (1) assumes that Euclid's Def.1.8. defines angle as a relation (of "inclination" of two intersecting lines) and (2) provides an argument intended to show that this definition is not appropriate.

Proclus' commentary on Def.1.8.

[I]f the angle is an inclination [as in Euclid's Def. 1.8.] and in general belongs to the class of relations, it will follow that, when the inclination is one, there is one angle and not more. For if the angle is nothing other than a relation between lines or between planes, how could there be one relation but many angles?

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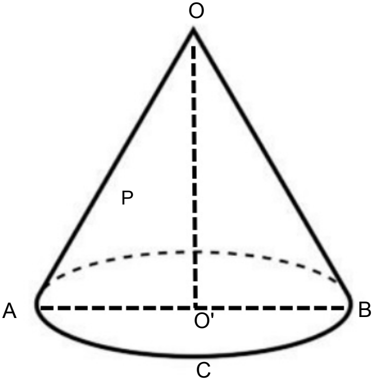
If you imagine a cone cut by a triangle from apex to base, you will see one inclination at the apex of the half-cone, that of the sides of the triangle, but two separate angles, one the angle on the plane of the triangle, the other on the mixed surface of the cone; and both of these angles are contained by the above-mentioned two lines. The relation of these lines, then, did not make the angle.

Proclus' cone construction

The same relation of inclination between straight lines OA, OB produces two different angles $\angle AOB_P$ and $\angle AOB_C$.



Proclus' argument in Vladimir' terms

Angle is a *structure* over pair OA, OB but not a relation that holds over this pair of terms.

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Angles $\angle AOB_P$ and $\angle AOB_C$ link lines OA, OB in two different ways.

HoTT

In HoTT/UF we have a similar situation: terms (points) A, B are identified via different non-homotopical paths p, q; homotopical paths can be identified via non-equivalent path homotopies, and so on up to the infinite hierarchy of homotopy levels.

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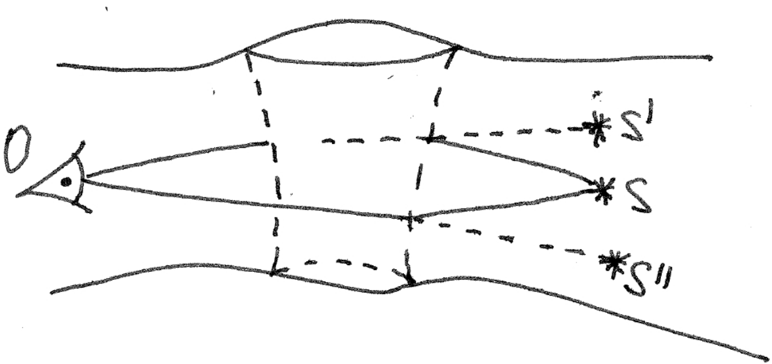
Example: higher identity types : $q, p : A =_S B$, $h_1, h_2 : q =_{A =_S B} p$, ...

Identity is a multi-level structure but not a mere relation! In HoTT this feature of identity predicate is transferred ("transported") to all predicates (including non-monadic ones usually called and understood as relations).

Example: hypothetical gravitational lensing around a wormhole in spacetime



Is Proclus' cone construction relevant to Def.1.8.?

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Proclus' argument can be saved, nevertheless, with the following additional argument: since (two-dimensional non-plane) angle cannot be defined as a relation of lines in the solid (3D) geometry, it should not be so defined in the plane (2D) geometry either — because the plane geometry is a special case of the solid geometry

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What is wrong then about the fact that in Proclus' cone construction two (in fact, three!) rather than just one angle fall under Euclid's definition?

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Proclus' assumption: Relational Uniqueness (**RU**):

An instance of a binary relation, (say $OA \bowtie OB$ where $\bowtie$ stands for "inclination"), is fully identified when its relata (OA and OB in our case) are identified.

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Corollary:

An instance of a binary relation is an (ordered) pair of items that hold that relation.

Grounding **RU**: Pairing in ZF

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For non-symmetric relations one needs a notion of ordered pair. The similar uniqueness principle for ordered pairs can be justified with ZF only via fixing a particular representation of ordered pairs with sets like

- $< X, Y > := \{\{X\}, \{X, Y\}\}$ (Kuratowski)
- $< X, Y > := \{\{X, 1\}, \{Y, 2\}\}$ (Haussdorf)
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- $\dots$

The identification of ordered pairs across their set-theoretic representations is a Benacerraf-style challenge.

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3. Using the cone construction show that pair (OA, OB) such that $OA \bowtie OB$ gives rise to two different angles $\angle AOB_P$ and $\angle AOB_C$.

Proclus' argument in the nutshell (classically)

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2. Assume **RU**.
3. Using the cone construction show that pair (OA, OB) such that $OA \bowtie OB$ gives rise to two different angles $\angle AOB_P$ and $\angle AOB_C$.
4. Conclude that angle cannot be soundly defined as pair lines such as (OA, OB) that stand in a certain relation (such as Euclid's relation of inclination $OA \bowtie OB$).

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In what follows I provide an alternative *constructive* reconstruction of Proclus' argument, which is based on constructive logical principles used in HoTT.

The truth of a proposition and the validity of a judgement (after Martin-Löf)

$OA \bowtie OB$ is a *proposition*, $\angle AOB : OA \bowtie OB$ is a *judgement* where $\angle AOB$ *witnesses* (proves) the fact that relation $OA \bowtie OB$ holds (and hence proposition $OA \bowtie OB$ is true).

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In the constructive logic one does not assume that propositions have their truth-values independently of one's reasoning. A proposition qualifies as *true* if it has a proof. It qualifies as *false* if there is a proof that it has no proof. A proposition may remain undecided and/or be provably undecidable.

Constructive version of **RU**

RUC:

Proposition P (in particular, one of the form xRy) has *at most one* proof (aka witness).

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4. Using the cone construction show that the pair (OA, OB) gives rise to two different angles $\angle AOB_P$ and $\angle AOB_C$ none of which can be identified with the hypothetical angle $\angle AOB$.
5. Conclude using **RUC** that angles cannot be defined (and identified) as *the* witnesses of a proposition of the form $X \bowtie Y$. Thus angle cannot be defined as a relation (between its sides).

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In order to reply to this straightforward objection, one needs to be more specific about the concept of proposition. In HoTT there is a convention according to wich only types with at most one term are called by the name of proposition (or *mere* proposition).

Homotopical hierarchy of types

The idea behind this terminological choice is that proposition P with set S is a higher-order *structure* that should be distinguished from the proposition itself. P can be obtained from S by identifying all its elements if there are some, and leaving it empty otherwise. So we get constructive counterparts of classical truth-values: a proposition either has a proof (i.e., is constructively true), or (provably) has no proof (i.e., is constructively false).

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Then come *2-groupoids* interpreted as above plus distinguishable homotopies (up to 2-homotopies), and so on up along the homotopical ladder.

Following Proclus: Is (group) isomorphism a relation?

No, because given a pair of groups G_1, G_2 one can, generally, identify more than one isomorphism of the form $i : G_1 \xrightarrow{\sim} G_2$.

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The concept of iso-relation is a *truncated version* of that of iso-map (or iso-cell). In this case the truncation amounts to collapsing of all iso-maps (iso-cells) of a given type into a single term (object).

Following Proclus: Is (group) isomorphism a relation (or rather a map)?

Suppose we have an algorithm A that for a given pair of groups G_1, G_2 either effectively computes an iso-map $i : G_1 \xrightarrow{\sim} G_2$ or proves that such an iso-map does not exist. Thus A decides whether $G_1 \sim G_2$ holds by constructing an iso-map. Does judgement $i : G_1 \xrightarrow{\sim} G_2$ fully convey us the "constructive meaning" of isomorphism?

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Remark: The isomorphism problem is known to be algorithmically *undecidable* in the general case. (The decidability problem was formulated by M. Dehn in 1911 and solved by P.S. Novikov in 1955). The group isomorphism problem is, however, decidable for some special classes of groups.

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Since judgment $i : G_1 \sim G_2$ requires performing iso-map i, knowing that $G_1 \xrightarrow{\sim} G_2$ and knowing-how to perform i amount to the same. So one may be inclined to answer the above question positively.

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This judgement does not fully convey us the meaning of the concept of iso-map, however, because it doesn't tell us how to distinguish one such map from another one.

Cf. Quine's motto *No entity without identity*. The type of iso-maps of the form $i : G_1 \sim G_2$ is a set-level type but not a propositional type .

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Vladimir's question of whether or not the 'world' can be fully accounted for in terms of relations, i.e., mathematical structures in the usual sense of the word, is tantamount to the question of whether or not it can be fully described with (bi-valuated) propositions. Vladimir suggests the negative answer.

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In many cases higher-order structures can be represented as (or by?) set-level structures. For example, a non-trivial groupoid can be given a (two-sorted) propositional description and presented as a set-level structure (as a multi-graph with a partial composition operation defined on its arrows).

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Open question: are similar set-level axiomatic representations available for all n-groupoids up to ∞-groupoids?

Whether or not the set-level reduction is always possible Vladimir was motivated by the idea of developing mathematics beyond the set level. As an example of such mathematics Vladimir could point to Motive theory and other areas of Algebraic Geometry where the language of higher categories seems appropriate (whether or not it is dispensable).

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The issue of pertinence of higher-order structures in mathematics has practical aspects: consider the recent developments in the Lean community, which since 2020 (or even earlier) has largely abandoned the ncat approaches and ideas and returned to classical "flat" formalisms (Kevin Bazzard).

Vladimir Voevodsky on the concept of mathematical structure in his letter exchange with Andrei Rodin

to appear in Springer (Synthese Library) volume *Mathematicians at Work: Empirically informed Philosophy of Mathematics* edited by Deborah Kant, José Antonio Perez-Escobar, Deniz Sarikaya, and Mira Sarikaya.

Preprint: https://arxiv.org/abs/2409.02935

Thank You!