

The Long History of (De)Formalisation of/in Mathematics

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SPHERE

23 February 2026

- 1 Formalisation from Euclid to Descartes to Leibniz to Peano to Coquand
- 2 Deformalisation of mathematics from the Stone Age to Egyptian scribes to Euclid
- 3 Where are we, and where are we going?

Euclid: Pythagorean Theorem 1, ca 300 BC

μζ'.

Ἐν τοῖς ὀρθογωνίαις τριγώνοις τὸ ἀπὸ τῆς τῆν ὀρθῆν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τῆν ὀρθῆν γωνίαν περιεχουσῶν πλευρῶν τετράγωνοις.

Ἐστω τρίγωνον ὀρθογώνιον τὸ $AB\Gamma$ ὀρθῆν ἔχον τῆν ὑπὸ $BA\Gamma$ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς $B\Gamma$ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν BA , $A\Gamma$ τετράγωνοις.

Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς $B\Gamma$ τετράγωνον τὸ $B\Delta E\Gamma$, ἀπὸ δὲ τῶν BA , $A\Gamma$ τὰ HB , $\Theta\Gamma$, καὶ διὰ τοῦ A ὀπιότερα τῶν $B\Delta$, ΓE παράλληλος ἦχθω ἡ AA' · καὶ ἐπεξεύχθωσαν αἱ $A\Delta$, $Z\Gamma$. καὶ ἐπεὶ ὀρθή ἐστὶν ἑκατέρω τῶν ὑπὸ $BA\Gamma$, BAH γωνιῶν, πρὸς δὴ τινὶ εὐθείᾳ τῇ BA καὶ τῷ πρὸς αὐτῇ σημείω τῷ A δύο εὐθεῖαι αἱ $A\Gamma$, AH μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας ἴσους ὀρθαῖς ἴσας ποιῶσιν ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓA τῇ AH . διὰ τὰ αὐτὰ δὴ καὶ ἡ BA τῇ $A\Theta$ ἐστὶν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ $\Delta B\Gamma$ γωνία τῇ ὑπὸ ZBA · ὀρθῆ γὰρ ἑκατέρω κοινῇ προσκείμεθω ἡ ὑπὸ $AB\Gamma$ · ὅλη ἄρα ἡ ὑπὸ ΔBA ὅλη τῇ ὑπὸ $ZB\Gamma$ ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔB τῇ $B\Gamma$, ἡ δὲ ZB τῇ BA , δύο δὴ αἱ ΔB , BA δύο ταῖς ZB , $B\Gamma$ ἴσαι εἰσὶν ἑκατέρω ἑκατέρω· καὶ γωνία

Proposition 47

In a right-angled triangle, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-angle.

Let ABC be a right-angled triangle having the right-angle BAC . I say that the square on BC is equal to the (sum of the) squares on BA and AC .

For let the square $BDEC$ have been described on BC , and (the squares) GB and HC on AB and AC (respectively) [Prop. 1.46]. And let AL have been drawn through point A parallel to either of BD or CE [Prop. 1.31]. And let AD and FC have been joined. And since angles BAC and BAG are each right-angles, then two straight-lines AC and AG , not lying on the same side, make the (sum of the) adjacent angles equal to two right-angles at the same point A on some straight-line BA . Thus, CA is straight-on to AG [Prop. 1.14]. So, for the same (reasons), BA is also straight-on to AH . And since angle DBC is equal to FBA , for (they are) both right-angles, let ABC have been added to both. Thus, the whole (angle) DBA is equal to the whole (angle) FBC . And since DB is equal to BC , and FB to BA ,

Euclid: Pythagorean Theorem 2, ca 300 BC

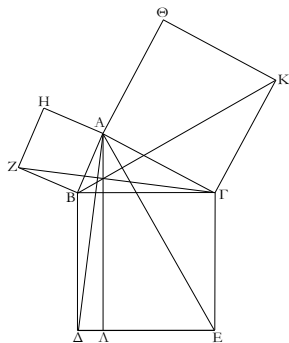
ΣΤΟΙΧΕΙΩΝ α'.

ELEMENTS BOOK 1

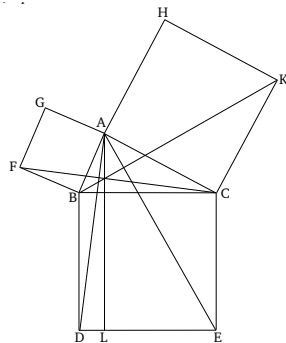
ἡ ὑπὸ ΔΒΑ γωνία τῆ ὑπὸ ΖΒΓ ἴση· βάσις ἄρα ἡ ΑΔ
βάσει τῆ ΖΓ [ἐστίν] ἴση, καὶ τὸ ΑΒΔ τρίγωνον τῷ ΖΒΓ
τρίγωνῳ ἐστὶν ἴσον καὶ [ἐστὶ] τοῦ μὲν ΑΒΔ τρίγωνου
διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν
αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις
ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τρίγωνου διπλάσιον τὸ ΗΒ
τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν
ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ.
[τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν] ἴσον ἄρα
ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ.
ὁμοίως δὴ ἐπιζευγυμένων τῶν ΑΕ, ΒΚ δειχθήσεται
καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ·
ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυοῖ ταῖς ΗΒ, ΘΓ τε-
τραγώνοις ἴσον ἐστίν. καὶ ἐπει τὸ μὲν ΒΔΕΓ τετράγωνον
ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ,
ΑΓ· τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ
τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.

the two (straight-lines) DB , BA are equal to the two
(straight-lines) CB , BF ,¹ respectively. And angle DBA
(is) equal to angle FBC . Thus, the base AD [is] equal
to the base FC , and the triangle ABD is equal to the
triangle FBC [Prop. 1.4]. And parallelogram BL [is]
double (the area) of triangle ABD . For they have the
same base, BD , and are between the same parallels, BD
and AL [Prop. 1.41]. And parallelogram GB is double
(the area) of triangle FBC . For again they have the
same base, FB , and are between the same parallels, FB
and GC [Prop. 1.41]. [And the doubles of equal things
are equal to one another.]² Thus, the parallelogram BL
is also equal to the square GB . So, similarly, AE and
 BK being joined, the parallelogram CL can be shown
(to be) equal to the square HC . Thus, the whole square
 $BDEC$ is equal to the (sum of the) two squares GB and
 HC . And the square $BDEC$ is described on BC , and
the (squares) GB and HC on BA and AC (respectively).
Thus, the square on the side BC is equal to the (sum of
the) squares on the sides BA and AC .

Euclid: Pythagorean Theorem 3, ca 300 BC



Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς
τὴν ὀρθὴν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον
ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχοσῶν
πλευρῶν τετραγώνοις· ὅπερ εἶδει δεῖξαι.



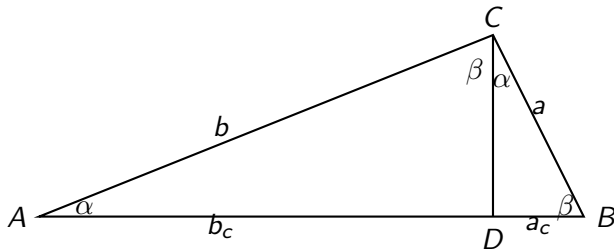
Thus, in a right-angled triangle, the square on the
side subtending the right-angle is equal to the (sum of
the) squares on the sides surrounding the right-[angle].
(Which is) the very thing it was required to show.

Pythagorean Theorem in a modern textbook

Theorem: Let a, b, c are sides of rectangular triangle and c its hypotenuse. Then $a^2 + b^2 = c^2$.

Proof: Let a_c, b_c be projections of a, b on the hypotenuse c . Then, by the equality of angles, $\triangle ABC \sim \triangle ACD \sim \triangle DCB$. It follows that

$$\frac{a}{c} = \frac{a_c}{a}, \quad \frac{b}{c} = \frac{b_c}{b} \Rightarrow a^2 = a_c c, \quad b^2 = b_c c \Rightarrow a^2 + b^2 = c^2 \blacksquare$$



Pythagorean Theorem in a modern textbook

The above modern textbook proof of the Pythagorean theorem involves ideas and notation first systematically introduced by René Descartes in 1637.

It does not essentially use any mathematical technique or knowledge obtained later on.

Descartes *La géométrie* 1637

GEOMETRIA,
à
RENATO DES CARTES

Anno 1637 Gallicè edita; postea autem
Vna cum NOTIS

FLORIMONDI DE BEAUVNE.

In Cùria Blesensi Consularii Regii, Gallicè conscriptis in
Latinam linguam versa, & Commentariis illustrata,

Operà atque studio

FRAN(CI)S(CI) à SCHOOTEN,
in Acad. Lugd. Batava Matheseos Professoris.

*Nunc demum ab eodem diligenter recognita, locupletioribus Commen-
tariis instructa, multisque egregiis accessionibus, præ ad ulteriorem
explicationem, quam ad ampliandam huius Geometriæ
excellentiâ faciendis, accommodata.*

Quorum omnium Catalogum pagina verè exhibet.



AMSTELODAMI,
Ex Typographia BLAVIANA, MDC LXXXIII.
Sumptibus Societatis.

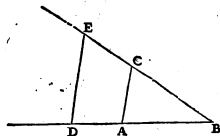


Descartes *La géométrie* 1637

2 GEOMETRIÆ

*Quomodo
Geometrice
fiat.
Multiplicatio,*

resve mediæ proportionales, quod idem est, quod radices Quadratæ, aut Cubicæ, &c. extractio. Neque enim hocse Arithmetices terminos, ut facilius intelligi possim, in Geometriam introducere verebor.

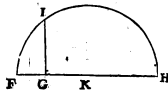


Sit, exempli gratiâ, A B unitas, oporteatque multiplicare B D per B C : jungo puncta A & C, ductaque D E parallelâ A C, erit B E productum hujus multiplicationis.

Divisio,

Vel si dividenda sit B E per B D, junctis punctis E & D, duco A C parallelam ipsi D E, eritque B C quociens hujus Divisionis.

*Extractio
radicis
Quadratæ.*



Vel denique si ex G H extrahere oporteat radicem Quadratam, adjungo ipsi in directum lineam rectam F G, quæ unitas est; divisâque F H bifariam in puncto K, centro K intervallo

FK seu KH describo circulum. quo facto, erit G I, quæ ex puncto G perpendicularis ducitur super F H usque ad I, radix quæsitâ.

Al-Khwarizmi, Algebra, ca 820 CE

(12)

the number of the roots must be halved. And know, that, when in a question belonging to this case you have halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of dirhems connected with the square, then the instance is impossible;* but if the product be equal to the dirhems by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition or subtraction.

In every instance where you have two squares, or more or less, reduce them to one entire square, † as I have explained under the first case.

Roots and Numbers are equal to Squares; ‡ for instance, “three roots and four of simple numbers are equal to a square.” Solution: Halve the roots; the moiety is one and a half. Multiply this by itself; the product is two and a quarter. Add this to the four; the sum is

* If in an equation, of the form $x^2 + a = bx$, $(\frac{b}{2})^2 < a$, the case supposed in the equation cannot happen. If $(\frac{b}{2})^2 = a$, then $x = \frac{b}{2}$

† $cx^2 + a = bx$ is to be reduced to $x^2 + \frac{a}{c} = \frac{b}{c}x$

‡ 3d case $cx^2 = bx + a$

Example $x^2 = 3x + 4$

$$\begin{aligned}x^2 &= \sqrt{[(\frac{3}{2})^2 + 4]} + \frac{3}{2} \\ &= \sqrt{(1\frac{1}{2})^2 + 4} + 1\frac{1}{2} \\ &= \sqrt{2\frac{1}{4} + 4} + 1\frac{1}{2} \\ &= \sqrt{6\frac{1}{4}} + 1\frac{1}{2} \\ &= 2\frac{1}{2} + 1\frac{1}{2} = 4\end{aligned}$$

Johann Faulhaber, *Miracula Arithmetica*, 1622

Ich setze den unbekandten Zahlen x & y A. und z B. dergestalt.

$\frac{1}{2} x^2 + x = A.$ Nota. mit x und $\frac{1}{2}$ muß man ein grosse
 $\frac{1}{2} z^2 + z = B.$ experiencz haben/waß man dieses gnug
 samb obseruiren will/te.

$\frac{1}{2} x^2 + x + A.$
 $\frac{1}{2} x^2 + z + A$
 $+ z = B + A.$

$\frac{1}{2} z^2 + z + B.$
 $\frac{1}{2} z^2 + x + B$
 $+ A = x + B + A.$

Auß diesem folgt.

Das $\frac{1}{2} x^2 + x + A$ gleich ist 10
 und $\frac{1}{2} z^2 + z + B$ gleich 4 NB.
 deßgleich $x + A$ gleich 8.

Leibniz, *De arte characteristica ad perficiendas scientias ratione nitentes*, 1688 (I)

Si daretur vel lingua a quaedam exacta (qualem quidam Adamicam vocant) vel saltem genus Scripturae vere philosophicae, qua notiones revocarentur ad Alphabetum quoddam cogitationum humanarum, omnia quae ex datis ratione assequi licet, inveniri possent, quodam genere calculi, perinde ac resolvuntur problemata Arithmeticae aut Geometriae.

If we could find either a certain exact language (some people call it *Adamic*) or at least a sort of truly philosophical Scripture, in which [fundamental] notions were represented in the form of Alphabet of human thought, then everything that can be obtained from these notions by reasoning will be obtainable with computations similar to those used for solving arithmetical and geometrical problems.

Leibniz, *De arte characteristica ad perficiendas scientias ratione nitentes*, 1688 (II)

[I]ta sentio nunquam temere controversias finiri neque sectis silentium imponi posse, nisi a ratiocinationibus complicatis ad calculos simplices, a vocabulis vagae incertaeque significationis, ad characteres determinatos revocemur.

Thus I feel that controversies can never be definitely settled, nor can silence be imposed, unless we manage to represent complicated reasoning by simple [syntactic] computations, and represent [*but note fully replace!* – A.R.] vague and uncertain words by well-determined characters..

Leibniz, *De arte characteristica ad perficiendas scientias ratione nitentes*, 1688 (III)

Quae vero facti sunt, et a fortuna vel casu pendent eatenus ad
artem inveniendi non pertinere manifestum est.

But those truths which depend on fortune or chance [i.e., which
are contingent], are clearly not to be found by this art of discovery.

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Commentary: Leibniz does not assume completeness of his projected logical calculus. But he might be nevertheless surprised to learn that some arithmetical truths are contingent.

Why Hilbert is missing

I leave now aside Hilbert's foundational project aiming at a *meta-mathematical* study of formal calculi rather than the introduction of formal methods in the everyday mathematical practice and mathematical education.

Giuseppe Peano, *Formulario* 1895

Formulario: How an ultimate formalisation looks like in practice.

III.

§ 1.

$a, b, c, k \in n.o$:

1. $0 \in na$.
2. $a \in n \times 1$.
3. $a \in na$.
4. $ab \in na$.
5. $a \in nb, b \in na, = a = \underline{+}b$.
6. $a \in nb, b \in nc.o. a \in nc$.
7. $a, b \in nc.o. a + b, a - b \in nc$.
8. $a \in nb.o. ac \in nbc$.
9. $\triangleright .o. ac \in nb$.
10. $a \in b + nk, b \in c + nk.o. a \in c + nk$.
11. $\triangleright .a' \in b' + nk.o. a + a' \in b + b' + nk$.
12. $\triangleright .o. ca \in cb + nk$.
13. $\triangleright .a' \in b' + nk.o. aa' = bb' + nk$.
14. $\triangleright .o. a^m \in b^m + nk$.
15. $ca \in cb + nck.o. a \in b + nk$.

Giuseppe Peano, *Formulario* 1901

Deformalisation of *Formulario*: a historical precursor of MALINCA project:

* 4. Propositions primitives

·0 $N_0 \in \text{Cls}$ Pp ·1 $0 \in N_0$ Pp

·2 $a \in N_0 \Rightarrow a + \in N_0$ Pp

·3 $s \in \text{Cls} \cdot 0 \in s : x \in s \Rightarrow x + \in s \Rightarrow N_0 \subset s$ Pp

{ P·3 = Induct = « loi d'induction » }

{ PASCAL a.1654 t.3 p.298 :

« Premier lemme ... cette proposition se rencontre dans la seconde base...

Deuxième ... si cette proposition se trouve dans une base quelconque, elle se trouvera nécessairement dans la suivante.

D'où il se voit qu'elle est nécessairement dans toutes les bases. » }

Les idées primitives sont déterminées par les propositions primitives que nous venons d'énoncer et par les P6-2 P8-4, desquelles découlent toutes les P de l'Arithmétique.

Dans la lecture des propositions il convient de se rapprocher autant que possible du langage ordinaire. On lira les P4 p. ex. comme il suit:

·0 « N_0 est une classe » ·1 « à laquelle appartient 0 »

·2 « Tout nombre est suivi par un nombre. »

·3 « Soit s une classe; supposons que 0 appartienne à cette classe; et que toutes les fois qu'un individu appartient à cette classe, son suivant y appartienne aussi; alors tous les nombres appartiennent à cette classe. »

On appelle « principe d'induction », cette Pp. On peut aussi la lire : « Si une

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Lebombo bone, the estimated age is 42-43 Kyears

30 notches found on this bone may correspond to the Lunar month.



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30 notches found on this bone may correspond to the Lunar month.
Human *mathematical* scripts (or samplings) historically (by far)
predates *writing* (in the usual sense of graphical representation of
spoken language).

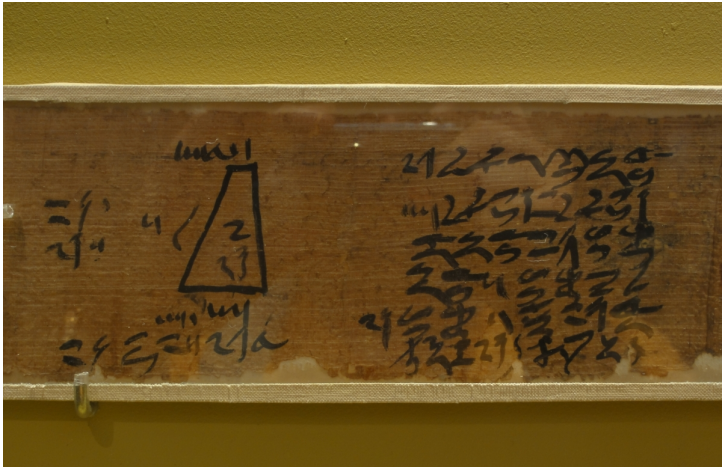


Clay accounting tokens from Tello (anc. Girsu), excavations Genouillac 1930-31, ca. 3500-2900 BC.



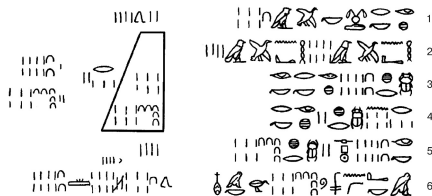
Moscow (Golenischev) papyrus, ca. 2000 BC

Combining a mathematical script with writing (a text in natural language) as a form of (meaning-giving?) *deformalisation*.



Moscow (Golenischev) papyrus, ca. 2000 BC

If you are told [problem]: a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. [Solution:] You are to square this 4; result 16. You are to double 4; result 8. You are to square this 2; result 4. You are to add the 16 and the 8 and the 4; result 28. You are to take $\frac{1}{3}$ of 6; result 2. You are to take 28 twice; result 56. See, it is of 56. You found it right.



Moscow (Golenischev) papyrus: commentary

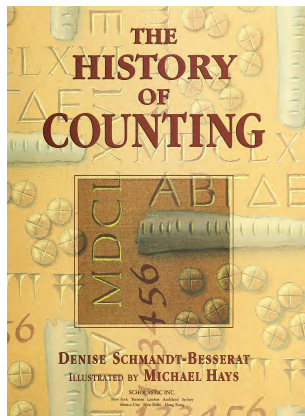
Text in the right column is written in Ancient Egyptian language using hieratic writing along with standard arithmetical symbols. It parallels the formal computation presented in the left column.

Moscow (Golenischev) papyrus: commentary

Text in the right column is written in Ancient Egyptian language using hieratic writing along with standard arithmetical symbols. It parallels the formal computation presented in the left column.

Why the author of the document opted for supporting the formal computation with a text in (the hieratically written) Egyptian?

Denise Schmandt-Besserat: all forms of early human writing derive from mathematical scripts.



Theoretical Mathematics: back to Euclid

It appears that the *theoretical* mathematics concerned with proof, reasoning and understanding — as we know it, in particular, via Euclid's *Elements* — would be impossible without using the natural language in a systematic way.

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Adamic Language

The idea of *Adamic Language* (or *Adamic Script*) in the form of logical calculus which directly reflects basic ontological structures of our universe and thus enables us to know things in this universe and to reason about them correctly is a anti-scientific myth, which underlies a good deal of (so called *Analytic Philosophy* in the 20th century).

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Leibniz himself referred to the notion of *Adamic Language* rather ironically, but two centuries later Bertrand Russell took the idea of *Adamic Script* quite seriously.

Russell on Peano

The Congress [of Philosophy that was held in Paris in 1900] was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his **notation**. . . .

Russell on Peano

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It became clear to me that his **notation** afforded an instrument of logical analysis such as I had been seeking for years.

[*Autobiography*, 1967]

Neo-Scholasticism

In the course of the 20th century a large part of the Analytic Philosophy developed into a new form of dogmatic medieval-style Scholasticism where Aristotle's syllogisms were replaced by the elements of the Classical First-Order Logic.

Józef Maria Bocheński (1902-1995)

Józef Bochenski's fully embraced and emphasised the conceptual ties of Analytic Philosophy to the Medieval Scholasticism. He believed that his contemporary Analytic Scholasticism was an appropriate antidote against his contemporary Marxism and the related doctrine of *Dialectical Materialism* promoted by Kremlin ideologues and their heirs in other countries of the Communist Block (including Poland where Bocheński was born).



An exit strategy

The recent booming development of new formal calculi in Mathematical Logic and Computer Science, and the implementation of many of those calculi in the form of executable programs, undermines the monopoly of the Classical FOL in its pretended role of *the* language of rational argument and thought (i.e., of the *Adamic Script*) and calls for a revision of our current ideas about formal calculi and their epistemic roles.

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What we need today is not yet another replacement of Classical FOL by some more fashionable formal calculus but rather a reconsidering of the relationships between formal calculi, natural languages and other related symbolic tools used in our scientific, educational, and cultural practices.

Open questions:

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- How formal calculi obtain meaning via the natural language and otherwise?
- What are criteria of the adequacy of a formalisation of some given informal content?
- (Dually:) What are criteria of the adequacy of a deformalisation (aka interpretation) of some given symbolic content?
- ...

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must be abandoned both in science and its philosophy.

I hope that our today's workshop will contribute towards this goal.

Thank you!