Kolmogorov's Calculus of Problems and its legacy in the constructive mathematics

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Plan:

1. BHK-interpretation
2. Kolmogorov on the (true) meaning of the Intuitionistic Logic (1932)
3. Negated Problems and Unsolvable Problems
4. The Intuitionistic appropriation of CP
5. Conclusion: the Legacy of CP beyond the BHK-interpretation

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- A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B;
- Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

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- 1990: A.S. Troelstra in "The Early History of Intuitionistic Logic": a historical dimension.

Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Kolmogorov and Alexandrov in 1930 in Germany

Popular perception of Kolmogorov's Calculus of Problems

https://ncatlab.org/nlab/show/BHK+interpretation

> *The idea of the [BHK] interpretation is clearly expressed in Kolmogorov (1932, p. 59), though rather briefly and in unusual terminology: Instead of propositions, Kolmogorov speaks of Aufgaben (Deutsch for "tasks", but here in the sense used in math classes where it means "exercises" or "mathematical problems") [...].*

Claims

Whatever *theoretical* value the notion of BHK interpretation may have, it is a *historical* misconception. Kolmogorov's interpretation of the intuitionistic propositional calculus differs essentially from Heyting's. The difference is not merely terminological.

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Kolmogorov's interpretation worths to be considered theoretically apart from the BHK interpretation.

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Kolmogorov 1932: the main purpose of the paper

Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [...] If we can reduce the solution of problem *b* to the solution of problem *a*, and the solution of problem *c* to the solution of problem *b*, then the solution of *c* can also be reduced to the solution of *a*.

Kolmogorov 1985: the author's commentary

Paper "On the interpretation of intuitionistic logic" was written [back in 1932] with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types — propositions and problems.

Heyting 1934

[After presenting his interpretation of the intuitionistic calculus.] Kolmogorov developed an akin [verwandten] idea which, however, goes beyond the former idea [i.e., beyond Heyting's interpretation] since it provides Heyting's calculus with a meaning that does not depend on the intuitionistic assumptions. (*Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*. Berlin: Springer)

Kolmogorov's Preface to Russian translation of Heyting 1934 published in 1936

We cannot agree with the intuitionists when they claim that mathematical objects are products of the constructive activity of our spirit. For us, mathematical objects are abstractions from the existing forms of reality, which are independent from our spirit.

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- In view of Heyting (who tries to implement Brouwer's views on mathematics) and Heyting's followers including Anne Sjerp Troelstra and Per Martin-Löf *problems and theorems are the same*;
- For Kolmogorov the traditional distinction between problems and theorems is fundamental. Kolmogorov's interpretation of the intuitionistic calculus concerns the former but not the later.

(Martin-Löf / Sambin 1984)

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . .] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead we simply identify them, that is, treat them as one and the same notion.

Historical/Biographical comment

It is remarkable that Per Martin-Löf philosophically followed Brouwer and Heyting but not Kolmogorov who was his doctoral advisor in Moscow University. Did PML consider at the time the Calculus of Problem as an *alternative* semantics for IL to the BHK semantics? Probably not.

CP and the Law of Excluded Middle

LEM for problems amounts to an universal method of solving *all* (well-posed) problems positively or negatively (i.e., by deriving a contradiction from the hypothesis that a positive solution exists). Since there is not such a method in view, LEM does not apply universally in this domain.

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This argument has no bearing on whether or not LEM applies in the domain of propositions. Kolmogorov believes that Brouwer's examples suggesting the negative answer are artificial and can be isolated and explained away.

Kolmogorov 1929

The law of excluded middle according to Brouwer could not be applied only to a certain kind of judgements, in which a theoretical statement is closely connected with construction of the object of the statement. Therefore, we may assume that Brouwer's ideas do not contradict the traditional logic, which has never before dealt with such judgements.

Kolmogorov 1932: concluding paragram

As to mathematics, it follows that the solution of a problem must be considered as an independent task along with the proof of theoretical propositions.

Kolmogorov 1932: Note added during the correction

This interpretation of intuitionistic logic is closely related to the ideas that Mr. Heyting developed in the last volume of *Erkenntnis* (1931); However, Heyting fails to distinguish clearly between propositions and problems [Aussagen und Ausgaben].

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Kolmogorov 1932: intuitionistic negation

By negation of problem a, in symbols $\neg a$, Kolmogorov understands 'the problem "to obtain a contradiction provided that the solution of a is given"', which is the standard intuitionistic negation used by Heyting. Kolmogorov's explanation of negation is followed by the following interesting footnote:

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Let us observe that $\neg a$ should not be understood as the problem 'to prove the unsolvability of [problem] a'. If one considers in general 'the unsolvability of a' as a well-defined concept, than one only obtains the theorem that from $\neg a$ follows the unsolvability of a, but not the converse. For example, if it were proven that the well-ordering of the continuum surpasses our abilities, one could still not claim that a contradiction follows from the existence of such a well-ordering.

Kolmogorov 1932: intuitionistic negation

Thierry Coquand interprets this remark as follows: it may be the case (and indeed is the case when $T = \mathrm{ZF}$) that the existence of well-ordering of the continuum (or the existence of some other mathematical construction) is not provable in the given theory T but is nevertheless consistent with this theory (since AC is independent of ZF)

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But Kolmogorov in the above quote doesn't say "it cannot be proved". He says "if it *were proven*" that the ordering of continuum surpasses constructive means assumed in a given theory.

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Notice the shift from quantifying over geometrical figures to quantifying over the *methods* of their construction.

Kolmogorov 1932: (ultra)intuitionistic critic of (intuitionistic) negation

The major result of the intuitionistic criticism of negated propositions should be formulated in the following simple way: in general it is meaningless to consider the negation of a general proposition as a definite proposition.

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But then the subject of the intuitionistic logic disappears, since the law of excluded middle becomes true for all propositions whose negation is meaningful.

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Cf. the "negationless intuitionistic mathematics" by G.F.C. Griss 1948

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Why this change of the narrative? Was it just a matter of oblivion?

Possible explanation:

During the time span between 1934 and 1958 Heyting contributed a very significant effort into the project of developing the *intuitionisitic mathematics*. Main results of this work are summarised in *Heyting 1956*, which includes chapters on (the intuitionsitc versions of) Number Theory, Real Analysis, Algebra, Geometry, Measure Theory and Integration Theory, elements of Functional Analysis (Hilbert spaces), and, finally, Logic (in this order).

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Unlike Heyting Kolmogorov was not interested by developing anything like the "intuitionistic mathematics", and I didn't even see this combination of words in his writings.

Troelstra and van Dalen

Anne Sjerp Troelstra and Dirk van Dalen were Arend Heyting's students who further developed the same project under his supervision. Troelstra's Ph.D. thesis (1966) is on the "Intuitionistic General Topolgy" ; van Dalen's thesis (1963) is on the "Intuitionistic Plane Projective Geometry".

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Troelstra&van Dalen's work in Logic including their notion of BHK-interpretation (1988) is a part of this larger project.

van Dalen 1979

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This is a judgement of someone who follows in Heyting's steps and is ready to acknowledge any contribution to *Brouwer-Heyting's* project of developing the "intuitionistic mathematics" but is not interested in studying competing projects like Kolmogorov's.

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Van Dalen is an actor directly involved in Brouwer-Heyting's project but not a non-partisan historian who wants to understand a plethora of competing projects in the foundations of mathematics. I'm trying here to be such a disengaged historian and represent Kolmogorov's project in its own terms.

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In this setting to prove or disprove a proposition is a *specific* sort of problem among higher (h)level problems that are *not* of the same sort.

Are problems and theorems one and the same thing after all? (contd)

There are also other theoretical settings and practical contexts where making the distinction between problems and theorems makes perfect sense and appears indispensable – like in the works of Kolmogorov's student Albert Abramovich Muchnik (1934-2019) on Turing Degrees.

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The mutual epistemic roles of propositions and higher-order constructions in today's mathematics needs to be better understood and better articulated in the current mathematical practices including computer-assisted practices (automated proof verification and proof search).

Conclusion

Kolmogorov's Calculus of Problems worths both to be correctly accounted for in non-partisan historical terms *and* used as a motivation for a future work quite independently from the "BHK" interpretation.

THANKS!

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Kolmogorov's Calculus of Problems and Its Legacy

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