Logic and Geometry in Topos theory and in Homotopy Type theory

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Claim 1

The received notion of axiomatic theory stemming from Hilbert 1899 is not adequate to the recent successful practice of axiomatizing mathematical theories. In particular, the axiomatic architecture of the elementary Topos theory and (more obviously) of the Homotopy type theory (HoTT) does not fit into the standard Hilbertian pattern of formal axiomatic theory.

Claim 2

At the same time, Topos theory and HoTT both fall under a more general and in many respects more traditional notion of axiomatic theory stemming from Euclid's *Elements*. Using some elements of HoTT I shall formulate the notion of constructive axiomatic theory in precise terms.

Claim 3

This broader notion of axiomatic theory, which I call after Hilbert and Bernays *constructive*, is more suitable for being used as a formal framework in physics and other sciences than the received notion.

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- The standard *statement view* of theories (SV):
  Axiomatic theory $< T, A, \Rightarrow >$ is a set T of *propositions* with a distinguished subset $A \subset T$ of *axioms* and relation $\Rightarrow$ of *deducibility*, which generates T from A.

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- Fixity of logical semantics (Fixity of Logic: FL):
  A rigid distinction between logical and non-logical syntactic terms; semantic values of logical terms are fixed beforehand while semantical values of non-logical terms vary from one model of a given theory to another model.

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- SV is to be contrasted against the *Non-Statement View* of theories (NSV) put forward by P. Suppes and supported by B. van Fraassen and others. According to NSV a theory is not a set of propositions but a class of *models*. As far as models are understood *á la* Tarski NSV is compatible with RAM at the price of distinguishing between semantic theories (= classes of models) and their syntactic representations. The notion of *constructive* axiomatic theory suggested by HoTT supports a stronger version of NSV, which is not

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- An axiomatic theory qualifies as *formal* when it displays such a contrast between the fixed logical part and the variable non-logical part. The *form* in question is a *logical* form.
- The distinction between logical and non-logical terms is typically made *ad hoc* but it calls for a general criterion of logicality.

Now I'm going to provide some examples of modern axiomatic theories, which violate SV or FL or both.

Elementary Theory of Category of Sets (ETCS)

Lawvere 1964
The Idea (back to von Neumann in late 1920-ies): use functions and their composition instead of sets and the primitive membership relation $\in$ used in the ZF and its likes
Remark: The project as it stands is fully compatible with RAM; the proposed deviation from the standard approach amounts to a new choice of primitives.

ETCS 1: ETAC

Elementary Theory of Abstract Categories (Eilenberg - MacLane)

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i, j = 0, 1$
- E2) $(\Gamma(x, y; u) \wedge \Gamma(x, y; u')) \Rightarrow u = u'$
- E3) $\exists u \Gamma(x, y; u) \Leftrightarrow \Delta_1(x) = \Delta_0(y)$
- E4) $\Gamma(x, y; u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x), x; x) \wedge \Gamma(x, \Delta_1(x); x)$
- E6) $(\Gamma(x, y; u) \wedge \Gamma(y, z; w) \wedge \Gamma(x, w; f) \wedge \Gamma(u, z; g)) \Rightarrow f = g$

E1)-E4): bookkeeping (syntax); 5): identity; 6): associativity

ETCS 2: Elementary Topos (anachronistically):

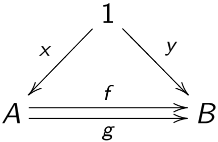
- finite limits;
- Cartesian closed (CCC): terminal object (1), binary products, exponentials;
- subobject classifier

$$\begin{array}{ccc} U & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle true} \\ X & \xrightarrow[\chi U]{} & 2 \end{array}$$

for all p there exists a unique χU that makes the square into a pullback

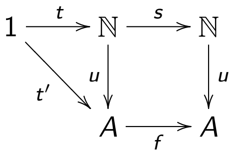
ETCS 3: well-pointedness

for all $f, g : A \rightarrow B$, if for all $x : 1 \rightarrow A$ $xf = xg = y$ then $f = g$



ETCS 4: NNO

Natural Numbers Object: for all t', f there exists unique u



ETCS 4: Axiom of Choice

Every epimorphism splits:
If $f : A \rightarrow B$ is epi then there exists mono $g : B \rightarrow A$ (called *section*) such that $gf = 1_B$

The idea of internal logic: CCC

- Lawvere 1969: CCC is a common structure shared by (1) the simply typed λ-calculus (Schönfinkel, Curry, Church) and (2) Hilbert-style (and Natural Deduction style) Deductive Systems (aka Proof Systems).
- In other words CCC is "the" structure captured by the *Curry-Howard correspondence* or *Curry-Howard isomorphism*
- The CCC structure is *internal* for *Set* BUT is more general: *Cat* (of all *small* categories) is another example; any *topos* is CCC.
- Lawvere's Hegelian understanding of this issue: CCC is *objective* while usual syntactic presentations or logical calculi are only *subjective*. While syntactic presentations lay out only *formal* foundations, CCC lays out a *conceptual* foundation.

Internalization of quantifiers: Hypodoctrines

Suppose that we have a one-place predicate (a property) P, which is meaningful on set Y, so that there is a subset P_Y of Y (in symbols $P_Y \subseteq Y$) such that for all $y \in Y$ $P(y)$ is true just in case $y \in P_Y$.
Define a new predicate R on X as follows: we say that for all $x \in X$ $R(x)$ is true when $f(x) \in P_Y$ and false otherwise. So we get subset $R_X \subseteq X$ such that for all $x \in X$ $R(x)$ is true just in case $x \in R_X$.
Let us assume in addition that every subset P_Y of Y is determined by some predicate P meaningful on Y. Then given morphism f from "universe" X to "universe" Y we get a way to associate with every subset P_Y (every part of universe Y) a subset R_X and, correspondingly, a way to associate with every predicate P meaningful on Y a certain predicate R meaningful on X.

Internalization of quantifiers: Substitution functor

Since subsets of given set Y form Boolean algebra $B(Y)$ we get a map between Boolean algebras:

$$f^* : B(Y) \longrightarrow B(X)$$

Since Boolean algebras themselves are categories f^* is a functor. For every proposition of form $P(y)$ where $y \in Y$ functor f^* takes some $x \in X$ such that $y = f(x)$ and produces a new proposition $P(f(x)) = R(x)$. Since it replaces y in $P(y)$ by $f(x) = y$ it is appropriate to call f^* *substitution* functor.

Existential Quantifier as adjoint

The *left* adjoint to the substitution functor f^* is functor

$$\exists_f : B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of X) into $P \in B(Y)$ (subset of Y) consisting of elements $y \in Y$, such that *there exists* some $x \in R$ such that $y = f(x)$; in (some more) symbols

$$\exists_f(R) = \{y | \exists x(y = f(x) \wedge x \in R)\}$$

In other words $\exists_f$ sends R into its *image* P under f. One can describe $\exists_f$ by saying that it transformes $R(x)$ into $P(y) = \exists_f x P'(x, y)$ and interpret $\exists_f$ as the usual existential quantifier

Universal Quantifier as adjoint

The *right* adjoint to the substitution functor f^* is functor

$$\forall_f : B(X) \longrightarrow B(Y)$$

which sends every subset R of X into subset P of Y defined as follows:

$$\forall_f(R) = \{y | \forall x(y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$.

Prehistory of Internal Logic

- Boole 1847, Venn 1882: propositional logic as algebra and mereology of (sub) classes (of a given universe of discourse); logical diagrams
- Tarski 1938 topological interpretation of Classical and Intuitionistic propositional logic

While in Boole, Venn and Tarski an internal treatment is given only the propositional logic Lawvere develops a similar approach to the 1st-order logic. There is a significant technical and conceptual advance between the two cases.

ETCS and RAM

In standard presentations as one by McLarty ETCS is presented according to the usual RAM standard; the feature highlighted above remains hidden. However the above analysis of ETCS suggests this idea: to use ETCS for introducing a relevant system of logic (Classical FOL) *along with* the theory in question (Set theory). This proposal is obviously *not* compatible with FL, and hence with RAM. Notice that it does not rely on the distinction between logical and non-logical terms in anything like its usual form.

Internal Logic and Internal Geometry

Consider an analogy between the concept of internal logic and that of internal geometry of a given (Riemanian) manifold. First, Gauss found the concept of internal curvature of a curve surface S, which is invariant w.r.t. how a given S is embedded into the outer 3D Euclidean space. Only later Riemann made a crucial step by suggesting that S can be thought of as a space on its own characterized by Gauss' invariants alone.

Internal Logic and Constructive Axiomatic Method (CAM)

I am aiming at a similar step in logic and its philosophical understanding. I assume that an axiomatic presentation of a theory does not need a fixed external logical framework just as the Riemanean geometry does not need any fixed external space, be it Euclidean or other. This does not mean that formal frameworks serving for multiple theories are useless. Different internal logics of different theories may be built from the same basic elements and on the same general principles.

Let us now see how the same axiomatic feature presents itself in the theory of Elementary Topos (ET), which can be described as the theory of Topos due to Grothendieck provided with an axiomatic form due to Lawvere. We shall see, in particular, how the concept of internal logic in this new context acquires a precise mathematical representation.

Quantifiers and Sheaves 1970

"The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos.

Quantifiers and Sheaves 1970

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry.

Internal Logic (more formally)

- Syntax (Mitchell-Bénabou): A sorted language L with lists of variables of every sort; sorts correspond to objects of the given topos; logical operations are compatible with usual operations with topos objects (produce, exponentiation, etc.)
- (External) Semantics (Kripke-Joyal): a formal satisfaction relation.

External vs. Internal View

Def. $f : A \to B$ is *epic* iff for all g, h $gf = hf$ implies $g = h$.
Def. Object T is *terminal* if for all object X there is unique arrow $X \to T$
Def. Arrow of the form $e : T \to A$ is called an *element of A*: $e =\in_T A$

Fact: $f : A \to B$ is epic iff it is internally onto: $y.B \vdash (\exists x.A)y = fx$

Warning: $y.B \vdash (\exists x.A)y = fx$ does not say that for each $y \in_T B$ there exists some $x \in_T A$ such that $y = fx$. Externally, f is epic but not necessarily *split* epic.

ET and RAM

Notice that L has a double semantics informally described by Lawvere in the above quote: it is both logical and geometrical. The same quote makes it evident that the internal logic of topos served Lawvere as a key for his axiomatization of Grothendieck's topos concept. However in standards RAM-style presentations such as one by McLarty this feature is once again hidden. The internal logic of a topos appears as an extra feature the logical and epistemic role of which remains rather unclear. It may be used for a further "reasoning in a given topos" but it plays no role - or at least no foundational role - in the construction of the base topos itself.

Open Question

How much of topos structure can be recovered from its own internal logic? How far the analogy between the topos logic and Riemanean geometry can be pushed forward?

The case of HoTT (which is a different framework!) suggests that the analogy between the internal logic of topos and internal geometry of manifold can be only partial - as far as *logic* is understood as usual as a system of rules for managing *propositions*. However if the concept of logic is extended to the effect that it may also include formal rules for non-propositional objects (cf. Kant's extended concept of logic) then the correspondence between logic and geometry can be made complete (in this different framework!) - in the sense that every logical rule admits a geometrical semantics and, vise versa, the geometrical theory in question admits an axiomatic presentation in purely logical terms without any "non-logical" term, rule or axiom.

In this case the distinction between logical and non-logical terms does not apply.

One may, however, recover the distinction between the logical and the non-logical content of HoTT in a non-trivial way by sticking to the standard "propositional" concept of logic; in this case HoTT allows one to establish the exact point at which the pure logic ends and a further geometry is brought in. So HoTT provides a support to Lawvere's idea according to which "logic is a special case of geometry".

HoTT: the idea

Types in MLTT are (informally!) modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of *n*-groupoids as higher homotopy groupoids of an appropriate topological space).

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x =_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER does not follow from other principles of MLTT (Streicher and Hoffman in 1990ies)!

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

Homotopy "model" of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x=_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x=_A y} y'$ is the space of all such homotopies
- ...

Cummulative Hierarchy of Homotopy Types

- 0-type: points in space with no (non-trivial) paths
- 1-type: points and paths in space with no (non-trivial) homotopies
- 2-type: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

Propositional reduction as truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

HoTT and RAM

Clearly HoTT violates both SV and FL!

Non-Propositional Content in Science

HoTT supports a strong version of NSF by providing a precise sense in which a theory, generally, does not reduce to the set of its propositions.

HoTT also supports a view according to which the non-propositional *Knowldge How* is a part of scientific and technical knowledge, which is at least as much important as the propositional *Knowledge That*). Observe that the core logical knowledge (= knowledge of logical inference) belongs to the former kind. My point is that the scientifically relevant *knowledge how* does not, generally, reduce to its logical variety.

Cassirer 1907 on Empirical Objects (contra Russell)

From the standpoint of logistics [= formal mathematics] the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions. Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences.

Mathematical Modeling in Science

The geometrical (homotopical) aspect of HoTT apparently makes this theory more apt for being used for mathematical modeling in Physics and other Sciences.

[E]xperience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. (Lawvere 1970 inspired by Hegel)

Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science. (Cassirer 1907)

Open Problem

It appears that we still miss a good replacement of Tarski's notion of model, which could work with HoTT and CAM more generally. Tarski's notion of satisfaction in its original does not make the whole job in such a context because it involves the concept of truth-evaluation and no alternative notion of model is universally accepted.

Axiomatic approaches to defining the satisfaction relation could probably be more relevant. The Model theory of HoTT is presently a subject of active research. This research revises basic conceptual issues such as the concept of model itself.

Conclusion 1

The constructive axiomatic architecture is rooted in history (Euclid) as well as in the recent successful practice of axiomatizing geometrical theories (ET, HoTT).

Conclusion 2

As the examples of ET and HoTT clearly demonstrate CAM involves a pattern of relationships between Logic and Geometry, which is quite unlike the corresponding pattern used in RAM. RAM-based axiomatic architecture leaves no room for a conceptual linking of geometrical principles to logical ones. Geometrical axioms appear here as very specific formal principles put on the top of logical principles and motivated solely by unspecified references to spatial experiences and intuitions. The CAM-based axiomatic architecture, in its turn, presents geometrical principles as a generalization of logical principles: in a CAM-based geometrical theory such as HoTT "logic is a special case of geometry". This non-standard pattern calls for a further epistemological analysis. Reconsidering some earlier views on logic and geometry such as

Conclusion 3

RAM proved effective as a very specific representational tool for meta-mathematical studies. But it appeared to be nearly useless for more general epistemic purposes, for which this method was originally designed or tentatively applied later. This includes the formal proof-checking, developing formal standards in scientific communication and education, developing a software for computer-based Knowledge Representation. Today's science and mathematics applies little of RAM-based methods and of logical methods more generally. Even in CS and software engineering the role of logical approaches appears rather modest.

Conclusion 3 (continued)

CAM already has a better performance and a better record in this respect. Its traditional informal version proved effective in mathematics (Euclid) and physics (Newton, Clausius). Today's proof-assistances such as COQ are CAM-based rather than RAM-based. There are reasons to expect that CAM-based logical methods (and perhaps HoTT more specifically) will apply in today's science and technology (including IT) more effectively that the standard RAM-based methods. In any event it is worth trying.

THANK YOU