Homotopy Type theory and the Bounds of Logic

Andrei Rodin

Russian Academy of Sciences and Saint-Petersburg State University

Overview 1

The standard Fregean concept of logical rule (as distinguished from, say, rules of chess or rules of building geometrical constructions) is that of a rule according to which certain *sentences* (judgements) are obtained from some other sentences (judgements).
The basic form of (categorical) *judgement* is

$$p \quad is \quad true$$

or in (more) symbols

$$\vdash p$$

where p is a *proposition*.

Example: *Modus ponens*

Frege 1918

"All sciences have truth as their goal; but logic is also concerned with it in a quite different way from this. It has much the same relation to truth as physics has to weight or heat."

Remark: It is not clear from this quote what the bounds of logic on Frege's conception. Are there parts of logic, which do not concern the issue of truth?

Overview 2

There are, however, some broader conceptions of logic (and corresponding conceptions of judgement) in which logical (?) rules are applied to *problems*, *intentions* etc.

Example: (1) BHK-semantics for the propositional intuitionistic logic, particularly Kolmogorov's version thereof; (2) MLTT

Overview 3

Homotopy Type theory (HoTT) puts these different informal logical semantics into an order: it specifies a particular fragment of "logic" (?), which is *propositional* in the above sense and another "higher" fragment where the logical (?) rules are applied directly to non-propositional (types of) objects. This order is enforced by robust mathematical theorems rather than by ideas *about* logic.

Warning: This division of HoTT does not coincide with the usual division of logical calculi into propositional calculi, 1-st order, 2-nd order and higher-order calculi.

Overview 4

The broader conception of logic supported by HoTT responds to some traditional concerns (Cassirer) about new logical developments (Frege, Peano, Russell) in the beginning of the 20-th century. It is a strong candidate for serving as a formal framework for Knowledge Representation systems of a new generation.

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

Remark

Only judgements of form (iii) can be interpreted as affirmed propositions.

MLTT : philosophical background

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

Martin-Löf 1983: Proposition

"Classical" notion of proposition as truth-value is rejected and replaced by the "intuitionistic" one:

"A proposition is defined by laying down what counts as a proof of the proposition."

"A proposition is true if it has a proof, that is , if a proof of it can be given."

semantic(s) of $t : T$ according to Martin-Löf 1983

- t is an element of set T
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x =_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher 1993l).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

HoTT: the idea

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of (informal) interpretations? NOT just that.

Homotopy "model" of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

Cummulative Hierarchy of Homotopy Types

- 0-type: points in space with no (non-trivial) paths
- 1-type: points and paths in space with no (non-trivial) homotopies
- 2-type: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types !

rather than

Proposition-as-Types

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

Propositional truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Example: dependent types

For propositional types:
If $\Gamma, x : X \vdash A : TYPE$, then $\Gamma \vdash X \to A : TYPE$

For non-propositional (higher) types: fibration

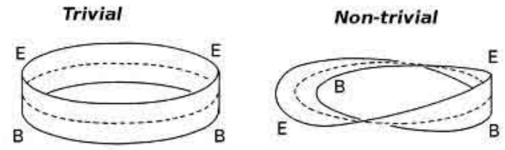

Рис.: Trivial and non-trivial fibration

Cassirer 1907 on Objecthood (contra Russell)

Here rises a problem that lies wholly outside the scope of "logistics" [..] According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions.

Cassirer 1907 on Objecthood (contra Russell)

Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences.

Mathematical Modeling in Science

The geometrical (homotopical) aspect of HoTT apparently makes this theory more apt for being used for mathematical modeling in Physics and other Sciences.

[E]xperience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. (Lawvere 1970 inspired by Hegel)

Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science. (Cassirer 1907)

The Semantic View of Theories (Reloaded)

HoTT and its model theory provides a novel notion of theory, which does notr reduce to a class of propositions but has a further higher-order non-propositional structure. The axiomatic basis of such a theory consists of a system of rules, which apply both at the propositional and non-propositional levels.

The Semantic View of Theories (Reloaded)

Such a broader concept of theory and its model better fits the colloquial counterparts of these notions in the scientific practice than the standard Tarskian notions. The main reason is that a typical scientific theory involves a lot of *procedural* content, which is used in modeling; such procedures may comprise but typically do not reduce to the procedures of logical inference (if by the logical inference one understands here a procedure which inputs and outputs sentences).

Open Problem

It appears that we still miss a good replacement of Tarski's notion of model, which could work with HoTT. Tarski's T-schema and his related notion of *satisfaction* in its original form does not make the whole job in such a context because it involves the concept of truth-evaluation and no alternative notion of model is universally accepted.

THANK YOU